The Cultural Impact

Of

Mathematics

THE CULTURAL IMPACT OF MATHEMATICS

UNIT II: MATHEMATICS AND MUSIC

INTRODUCTION

A few years ago most people would have greeted the idea of linking mathematics and music with considerable amazement.
It seemed incredible that music, the art which comes closest to evoking man's emotional states, should be related to mathematics, his most rational creation. Then, with the advent of computers and their much advertised ability to "compose" music,[1] it became generally recognized that music was accessible to mathematical analysis. People no longer question the fact that computers can be programmed to compose fugues like Bach or ballads like the Beatles.[2] Yet, it would be a gross distortion of history to believe mathematics and music had nothing to do with each other prior to the computer revolution.

The very fact that computers can be programmed to recreate historical styles of music and synthesize new ones points to an understandable logic in the organization of the elements of musical composition. This logic stretches back to the first composer who discovered that the twanging of his bow gave him more satisfaction than the release of its arrow.

Man orders the universe around him and does so in ways that bring him pleasure. Tone patterns are among the most pleasurable ways of ordering time. Indeed, so intense is this pleasure that music has evolved as one of the most complex and original of man's creations. It is no mere coincidence that the development of music in western civilization has coincided with the growth of mathematics. At certain critical points in their histories they have mutually nurtured each other. The earliest mathematics had to do with counting; the earliest music with the organization of sounds into rhythmic patterns. This fact was probably a stimulus for the comment by the mathematician and philosopher, Leibniz, that "Music is the pleasure that the human soul experiences from counting without being aware that it is counting."

As the ancient civilizations expanded so did mathematics and music. Both were treated as practical arts and maintained through tradition by the priestly castes. The ideas which worked in mathematics and the melodies and rhythms which pleased in music were the things that were preserved and handed down from generation to generation.

When at last, Pythagoras arrived on the scene (c. 540 B.C.) and conceived the structure of the universe as being ruled by number, music became a cornerstone of his philosophy. So it was that the origin of music as an experimental science coincided with the birth of mathematics as a deductive science.

Legend has it that Pythagoras was the first man to identify the mathematical basis of the "pleasure principle" in music. He found that a simple numerical correspondence existed between the musical tones that were preferred in the music of his time.[3] So important was this discovery that the Pythagoreans granted to mathematics the power "to make visible the invisible." From that time on, music theory became firmly entrenched in education as a branch of mathematics. It joined arithmetic, geometry, and astronomy as an essential part of the classical quadrivium which served as the basic curriculum for 2000 years.

Throughout the MiddleAges, the Pythagorean monochord[4] served as the essential pedagogical instrument in music education. Generations of singers and instrumentalists were instructed in proper ear training and intonation by this means.

It was through Pythagorean theory that the music of western civilization became standardized in numerical relationships, leading to the adoption of scales and modes which still persist in the composition of music today. Most particularly it provided a basis for the accurate construction and tuning of instruments.

With the purity of intervals so defined, a standard of reference was provided which finally permitted the development of polyphony and functional harmony, the most unique contributions of western civilization to the art of music.

Polyphony was a method for combining melodic lines simultaneously. By the time of the Renaissance it had evolved as a complex and tightly ordered system of composition. The equality of melodic lines was maintained by a mathematical structure of rhythmic proportions and a carefully formulated regulation of dissonance. Harmony, as such, occurred as accidental conjunctions of the intervals which vertically aligned themselves in the logical progressions of the melodic parts.

Around the year 1600, a change came about in musical practice that was to have the most profound impact on the course of musical development. What happened was that the vocal parts became particularized in the roles they carried in the musical fabric. The soprano became the vehicle for melody and the bass carried the weight of the harmony. The middle parts were subordinated to the role of fillers, rounding out the chordal structure. This new style which stressed the declamatory power of the harmonic content became known as homophony and it succeeded in adding a new dimension to music, analogous to the earlier Renaissance invention of perspective in painting.

Significantly, we find that at this same time Descartes and Fermat[5] were developing the foundations of analytic geometry. Thus, algebra and geometry were being forged into a unified language that would carry mathematical thought into the creation of a multi-dimensional calculus.

In both music and mathematics, the main problem to be solved at this time was that of directional motion. In music this was accomplished by the creation of tonality and in mathematics by the theory of gravity. The problems were essentially related. As the musicologist, Manfred Bukofzer, wrote: "Tonality may be defined as a system of chordal relations based on the attraction of a tonal center. This tonic formed the center of gravitation for the other chords. It is no mere metaphor if tonality is explained in terms of gravitation. Both tonality and gravitation were discoveries of the baroque period made at exactly the same time.,,"[6]

Newton developed the calculus as an essential tool for the presentation of his theory of gravitation. [7] From that point on the scientific advancement of musical theory was to be grounded in this mathematical discipline. It was finally to culminate in a complete explanation of vibrating mechanisms (which produce sounds) and Fourier's ultimate theorem for harmonic analysis of sounds.

Along the way, the development of logarithms (1614) provided Mersenne (1635) with the proper tool for calculating the intervals in equal temperament. Not only did this permit the construction of harmonic instruments like the piano, but it also expanded tonality by encouraging free modulation to distant keys. [8]

For the past 400 years the physics of music has grown to encompass and influence almost every aspect of the art of music. By the time Helmholtz was to write his definitive work, Sensations of Tone, in the late 19th century, most of the important acoustical facts about music had been discovered. Helmholtz succeeded in placing the science of musical acoustics on a firm experimental foundation. With the discovery of electricity and its technological possibilities, measuring devices have been perfected to confirm the experimental hypotheses of this science.

These technological advances have had a tremendous effect on music. The standardization and perfection of musical instruments is a direct result of this recent knowledge. Furthermore, the recording and reproduction of sounds has made music universally accessible. And finally, we are beginning to realize the potentialities for musical composition through the synthesis of sounds by electronic means. All of these advances would have been incomprehensible without the great surge of mathematical knowledge which culminated in the 19th century and which is still being extended today.

As you proceed through this unit following the details of the interaction between music and mathematics, be aware of the subtleties of the interrelationship. Music is basically an inductive art. Innovations and stylistic changes arise out of the artistic vision of individuals who are immersed in the problems of finding new modes of expressing their own unique musical personalities.

In the development of every musical style experimentation precedes formulation. Thus, tonality was being exploited by many composers to control the direction of harmonic motion several years before Rameau codified its rules in his theoretical writings. Similarly, Schoenberg had to trace his own path through post-Wagnerian chromatic harmony to a series of experimental compositions which abandoned tonality before he could set down the rules for his twelve tone system.

Before our current period of musical history, composers were seldom conversant with the mathematical foundations of their art. Proceeding intuitively, they were able to develop logical systems of composition which are only now becoming accessible to mathematical analysis. Computer theory has made it possible to generalize historical musical styles in a mathematical form and give promise of synthesizing new ones. [9]

Mathematics has also helped to illuminate some of the psychologica and physiological processes involved in the perception of music. The marvelous structure of the human ear in combination with its neural path to the brain permits a most sensitive discrimination of the aural information it receives. The hearing sense allows for a definition of tone as distinguished from noise and what is even more important for musical perception, it can differentiate between tones which are received simultaneously. [10] Thus, harmony, which is dependent upon the functioning of tones when they are sounded together, is a physiological possibility.

Finally we must be thankful for the unique capacity of the human brain to recognize and store musical patterns. Without this capacity, the perception of melody, rhythm, and musical form would be impossible.

There have been many attempts since the time of Plato to explain why we respond to music as we do. Musicians, philosophers, aestheticians, and scientists have all engaged in the fascinating search to understand the psychology of the human response to music. Recent advances in acoustical theories of music [11] have begun to emphasize the importance of noise elements in the production and perception of musical sounds. Philosophically, this seems to say that the human aspect of music is a function of its aural imperfections. This infers that emotional effects in music are created by deliberate deviations from the written score. [12]

The point of view adopted in the present text is that one must understand the standards of reference or idealizations of a system first, before the attempt is made to account for transient effects and departures from the norm. This is not to deny the importance of such variables as factors in musical perception but rather to place them in proper perspective. No matter how detailed a musical score is, the realization in live performances will always be different. That is what makes music such an exciting human experience. Yet, the essential architecture of this experience must always emanate from the logical formulations written into the score by the composer. The interpretive art can be no more than a revelation of what is contained therein.

Acoustical theories of music which focus on the atomization of individual sounds and try to suggest that musical meaning originates from the interplay of noise and tone are somewhat guilty of myopia in the interpretation of their new-found discoveries. They tend to underestimate the importance in musical cognition of formal design, as revealed in harmonic,melodic, and rhythmic structures and their relationships.

No mathematical theory of music will ever be successful until it can logically account for the total succession of events that make up a musical composition. In music, as it is in physics, we still lack a "unified field theory" to explain its grand design.

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Reference Notes to Introduction

1. To say that computers "compose" music is an oversimplification. Human beings set the parameters and direct the machine to deliver the possibilities of tonal patterns within these parameters.
See Hiller and Isaacson.
2. Though many may question the esthetic significance of the product.
3. A detailed coverage of Pythagorean music theory is given in the succeeding chapters.
4. The monochord was an experimental device which had led to the basic ideas of Pythagorean music theory.
5. The stimulus of perspective in painting to the development of projective geometry by Desargues about this time is well covered in Kline, Ch. XI.

6. Bukofzer, p. 12.

7. It should be noted, however, that in his earliest presentation of the theory, Newton relied upon synthetic geometry for his deductive arguments in order to assure that his ideas would be intelligible to the scientific community of his time.

8. The most famous compositions employing the new system of equal temperament were J. S. Bach's "Well Tempered Clavichord."

9. See Hiller and Isaacson.

10. See Jeans, Science and Music, p. 245ff.

11. Winckel, Music, Sound, and Sensation, A Modern Exposition.

12. A good example of this are the so-called "blue" notes of Jazz. Or, for opera lovers, consider Caruso's sobbing end of phrases.