UNIT I
MATHEMATICS AND THE VISUAL ARTS

CHAPTER 1 - Geometric Frameworks: A Survey

In this chapter geometric forms will be surveyed to see how they have been exploited in works of art. Along the way some important mathematical ideas shall be covered that will prove useful in understanding how mathematics relates to other fields.

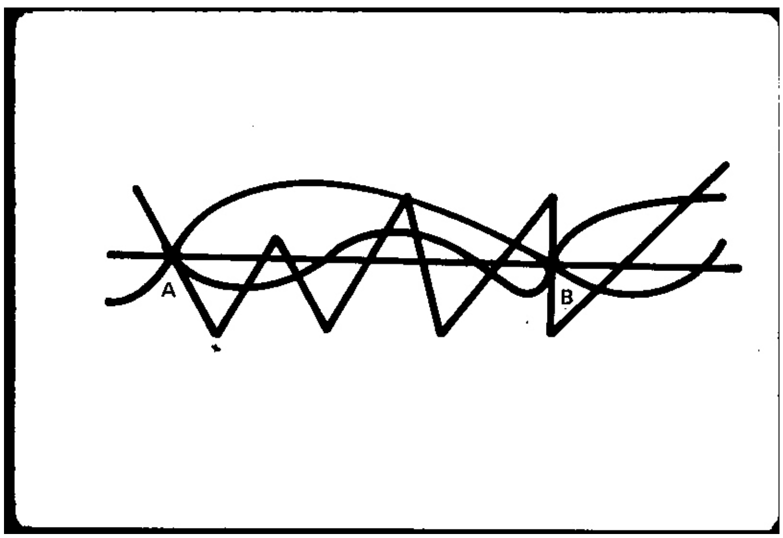
Although geometry has been of practical use to human beings since pre-historical times, the Greeks were the first people to draw abstract generalizations from geometric relationships. The culmination of 300 years of this development appears in The Elements by Euclid (c. 300 B.C.). From that time forward, the Elements would become the primary source for artists and craftsmen to learn the fundamentals of their discipline.

This pervading influence has lasted down to the present day and leading artists of our own time persist in using abstract geometric ideas in their works.

Lines

To begin then, in the manner of Euclid, we shall examine a postulate that introduces the concept of the straight line. Looking at Figure 1, two points are labeled A and B with several lines connecting them. Of all of these lines there is one and only one line which gives the shortest route from A to B. If A and B lie in a plane, this shortest route is called a straight line. The straight line is identified by the double arrow in the figure. If we consider that part of the straight line starting at A and terminating at B, we have what is called a line segment. If we consider only that part of the line which starts at B and moves to the right indefinitely, we have an example of a ray.

Figure 1



There are at least two things that lines do which are important in the visual arts; they set directions, and they define the boundaries of regions that enclose and divide the artistic space. The simplest directions are verticals and horizontals. Consider the Ulm Cathedral in Germany (1377-1492). http://en.wikipedia.org/wiki/Ulm_Cathedral

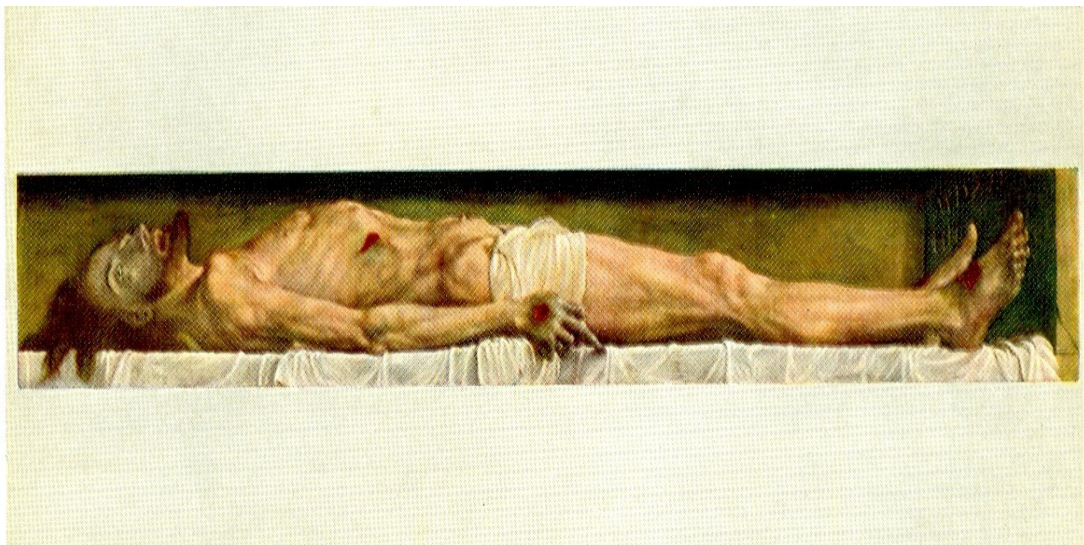
It can clearly be seen that the vertical elements predominate. This was a basic objective of Gothic architecture, to guide the eye of the beholder toward heaven. It can be noticed that the sets of vertical lines in the facade form a sequence which marches to the center and culminates in the pointed tower. In the world of signs that we live in, nothing says "this way only" more strongly than capping a straight line with an arrow. The Victory Tower from Iran (c. 1000 A.D.) prophetically captured the same shape and directional tendency of the rockets of our day.

http://en.wikipedia.org/wiki/Gunbad-e-Qabus

The horizontal line in art seldom performs in an active way as a directional indicator, unless it marks a series of intersecting planes and is actively pitted against vertical elements as shown in the famous example by Frank Lloyd Wright . http://en.wikipedia.org/wiki/Fallingwater

Usually it implies the antithesis of motion; a platform, a place to come to rest. It functions as a ground. The dramatic Entombment of Christ by Holbein (1497/8-1543) is an excellent example conveying this idea (Figure 2).

Figure 2



When the horizontal and vertical are combined the tension created by their opposition is felt. This is particularly true when the horizontal appears in an elevated position. The pathos engendered by scenes of the Crucifixion is enhanced by this structure .
http://en.wikipedia.org/wiki/Isenheim_Altarpiece

A balance can also be obtained between the horizontal and vertical that imparts a sense of order and stability. A fine example of this is the facade of an Elizabethan Inn to be found in Dartmouth, England (Figure 3). The cross-ribbed horizontals and verticals suggest a self contained society that is stable and permanent. The weary traveler coming to this hotel might well expect a substantial meal and a firm bed.

Figure 3

Lines that are neither parallel nor perpendicular to the boundaries of the artistic space are designated as oblique. These lines can function to control the perspective, creating a natural sensation of depth . They also

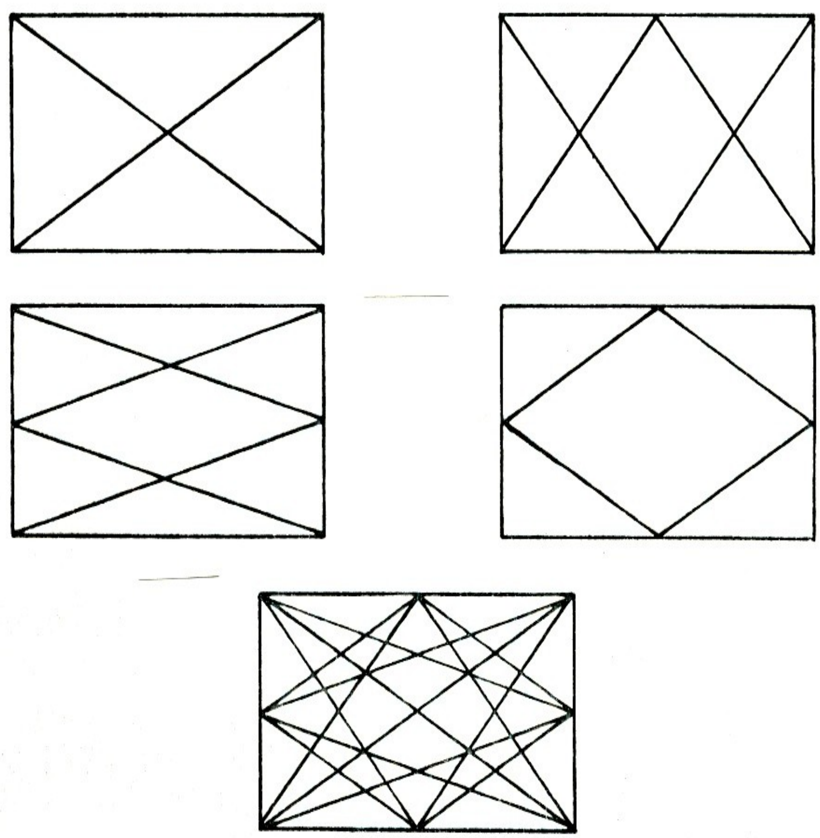
form frameworks that enable the artist to position figures and architectural objects in an organized manner. A viewer's perception of the scene can also be directed through so-called "lines of sight" that relate different parts of a composition.
http://en.wikipedia.org/wiki/Perspective_(graphical)

An exhaustive survey of such frameworks has been presented in a provocative book by the French artist, Charles Bouleau.[1]
He has analyzed many famous paintings from all periods and attempted to uncover the scaffolding which he believed served as structural foundations for the positioning of elements in them. He bases his analyses on the theoretical writings of the period and whatever actual drawings and sketches survive to substantiate his theories. His major thesis is that the space enclosing a work of art, whether it be two or three dimensional, contains its own inherent structural relationships that are independent of the composition it contains. To understand his approach, consider the following constructions. Since most paintings are contained in rectangular regions, oblique lines can most easily be drawn through the regions as diagonals that intersect at the center (Figure 4). Other lines can be drawn from the vertices to the midpoints of the opposite sides. These are called medians (Figure 5). In addition, the midpoints of each side can be connected to form a parallelogram (Figure 6 & 7). If these networks are joined together in one rectangle, a system is obtained which Bouleau labels the "armature of the rectangle." [2] It exists as an impersonal, objective necessity of the inner framework that emerges from the form itself and not from the artist's choice. According to Bouleau, he will make more or less

use of it, but will never be able to do without it entirely (Figure 8).

Figures 4-8

Building up the Armature of the Rectangle

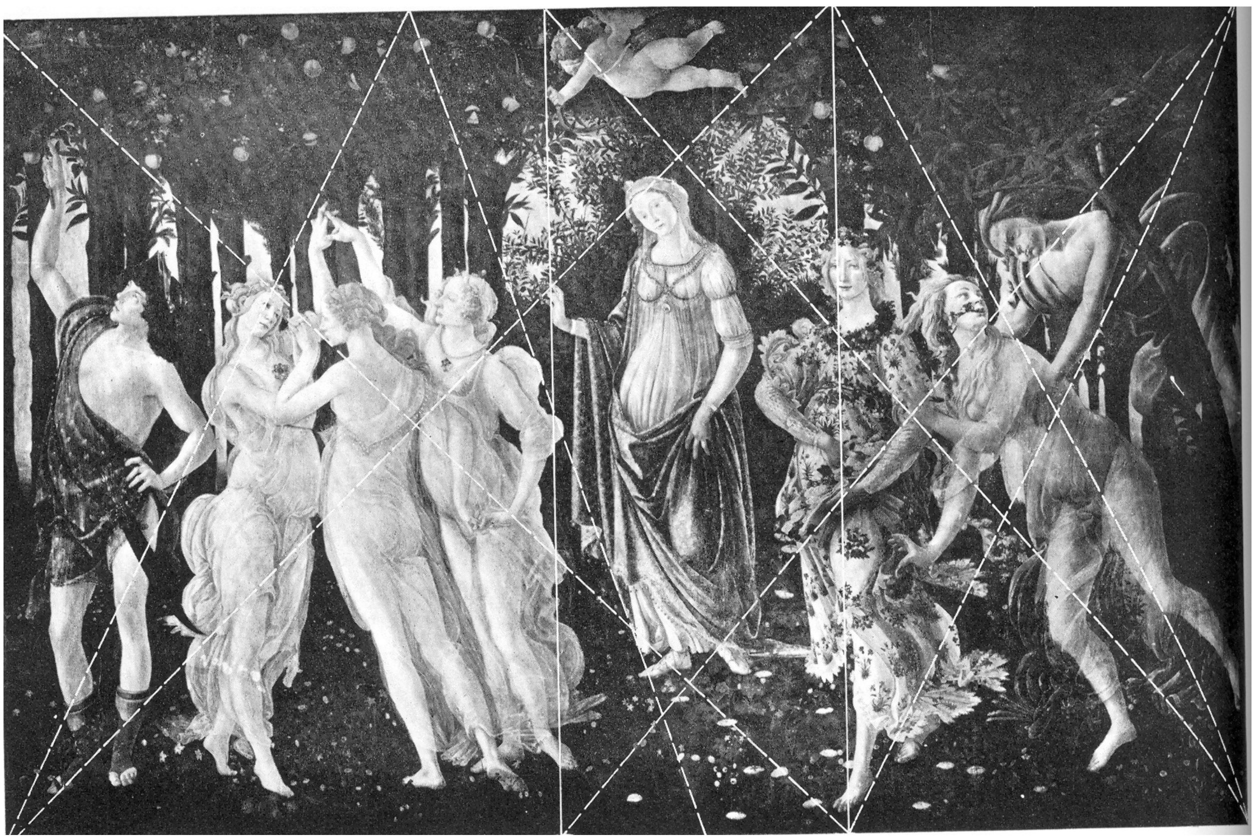


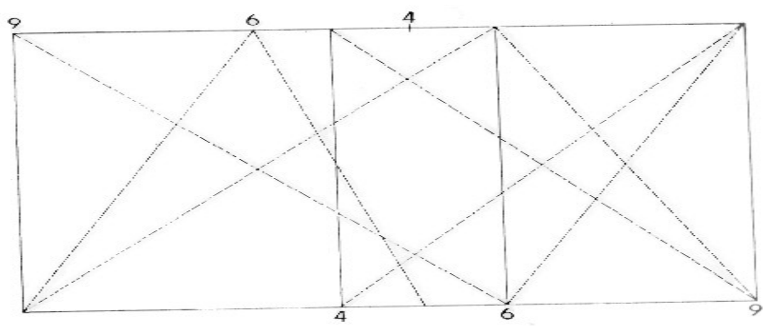
It is a well known historical fact that networks and grids were used in the construction of friezes, frescoes, stained glass windows, and other artistic surfaces from ancient times to at least the end of the Renaissance period, particularly to carry out the accepted rules of human proportions. [3] Whether or not the analyses that Bouleau provides are valid is still open to experimental verification and historical research. The major contribution of his work is to focus attention on the possible employment of such systems to position objects in relationship to each other and the dimensions of the artistic space they inhabit.

The proper rendering of perspective necessitated such a framework. In addition, Bouleau makes an excellent case for the idea that balance and proportion in a composition result from the integration of the artist's inner sense of order and his practical utilization of the linear matrices derived from the dimensions of the artistic space.

Frequently, however, non-symmetrical balancing was found desirable and this was achieved by the use of the musical ratios that defined the placement of tones in the musical scales or modes of Renaissance music.[5]

Bouleau reveals the beautiful organization that Botticelli (1446-1497) used in his famous work, Primavera .
http://en.wikipedia.org/wiki/File:Sandro_Botticelli_038.jpg

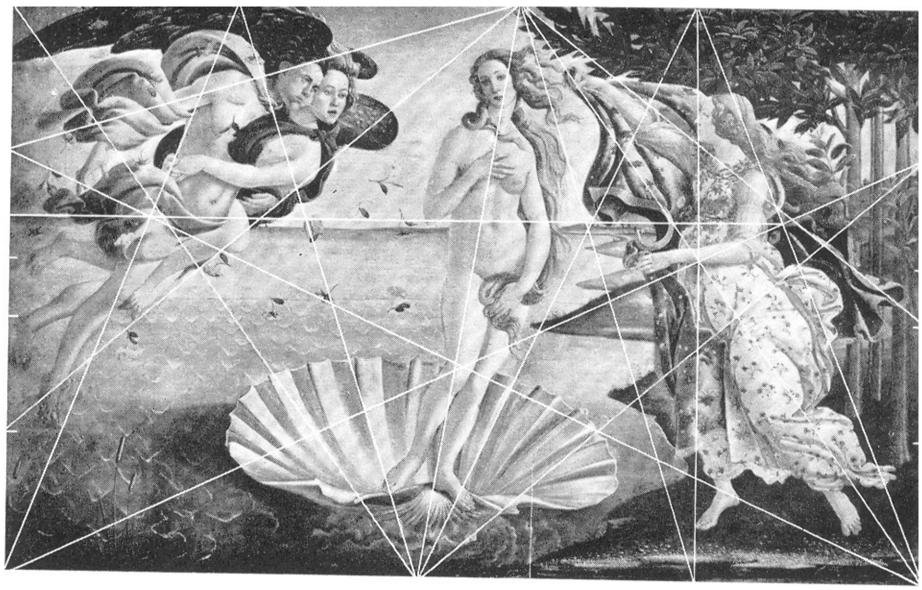


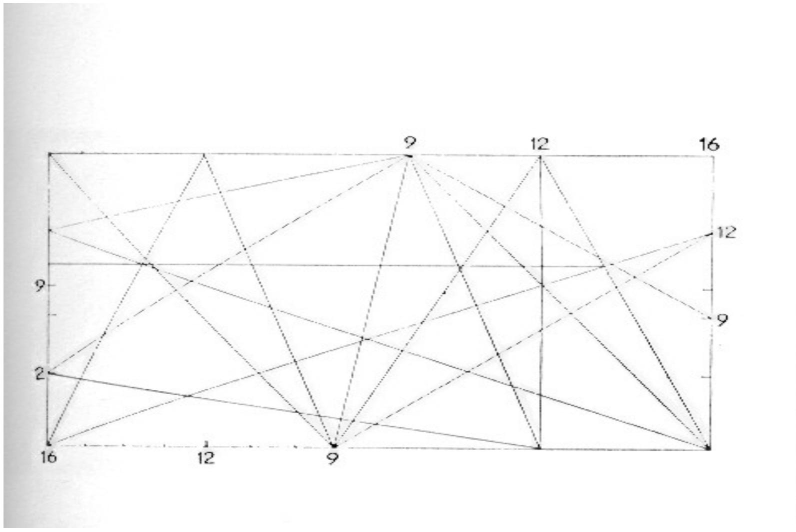


The ratios given are 4:6:9, a basic 2 to 3 ratio. Dividing the top and bottom of the canvas into nine equal parts, counting 1 to 9 from left to right, connect the 4 on the bottom to the 4 on top and the 6 on the bottom to the 6 on top. A division of the composition is produced which coincides with the grouping of the nine figures in the painting. On the left we find four figures. In the middle, going from 4 to 6 we find two figures. And finally, on the right there is a group of three figures in the space from 6 to 9.

As another example, Botticelli uses a variant of this scheme, according to Bouleau, to organize his equally famous, Birth of Venus. Here the framework is formed by the 9:12:16 ratio.

http://en.wikipedia.org/wiki/Botticelli%27s_Venus





The basic ratio is 3:4 and, as we shall see later, this determines what is known as the musical interval of a "perfect fourth"[6] In this framework the 9 on top is connected to the 9 on bottom, but the measure on the bottom is from right to left, thereby producing an oblique axis which defines the sculptural pose of the figure of Venus.

The employment of ratios derived from music in the visual arts of the Middle Ages and Renaissance is confirmed by many sources. Music was long considered to be a branch of science ever since the Pythagoreans of Ancient Greece had uncovered the mathematical laws of consonance. The architects of the Middle Ages employed the musical ratios in the building of their great cathedrals, viewing these immense structures as if their proportions would mirror cosmic harmonies. Renaissance artists adopted these concepts in the creation of their paintings and frescoes, striving to place their art on the same level of respect accorded to music as a science.

Bouleau's ideas on the frameworks underlying the composition of paintings will be amplified in later chapters of this unit. For now, we shall return to the survey of common geometric forms to be found in the visual arts.

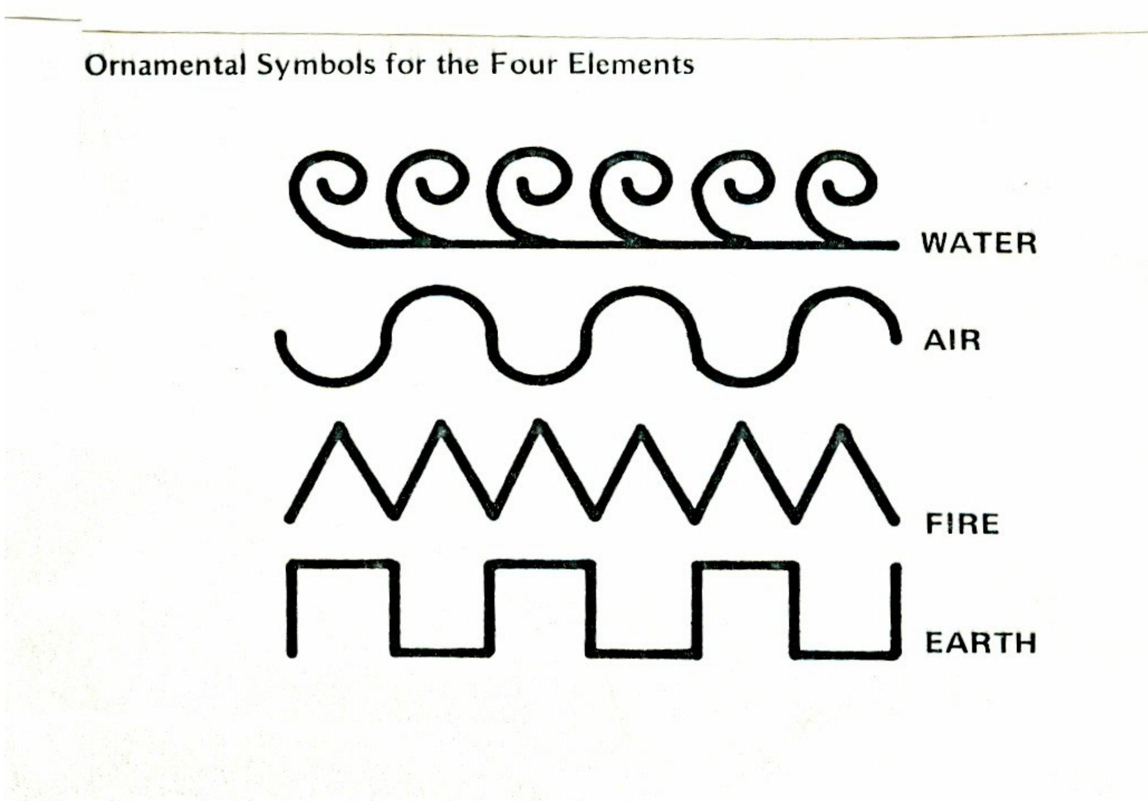
Polygons

Straight line segments can be organized into sets to form polygonal figures. Open polygonal designs are frequently found in primitive art and the early art of civilizations . In the traditional lexicon of ornamentation polygonal lines are symbols for fire and earth (Figure 9).

http://en.wikipedia.org/wiki/Geometric_Art#Early_Geometric_period

http://en.wikipedia.org/wiki/Mshatta_facade

Figure 9



Triangles

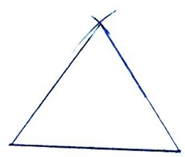
The simplest closed polygon is the triangle. Its unique structure combining horizontal, vertical, and oblique line segments has endowed it with extensive symbolic significance. Also, its identity with the number 3 has contributed to this mystical association. The identification of man with the divine is expressed in the emphasis on the triangular configuration of the head and shoulders and this design is often extended to his whole frame in art . As Cirlot[4] writes, "In its normal position with the apex uppermost it also symbolized fire and the aspiration of all things towards the higher unity." This is the meaning assigned to the triangular faces of the pyramid. The Greek word for fire is derived from "pyre" . The triangular positioning of figures enhances the emotional impact of Da Vinci's painting in the following link.http://en.wikipedia.org/wiki/The_Virgin_and_Child_with_St._Anne

A contemporary architectural example of the spiritual and spacial implications of the triangle are mirrored in the imaginative chapel to be found at the Air Force Academy in Colorado Springs. http://en.wikipedia.org/wiki/United_States_Air_Force_Academy_Cadet_Chapel

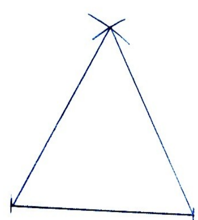
Triangles may be classified by the relationships exhibited by the line segments which compose their sides. In a scalene triangle, no two sides have the same measure (Figure 10). In an isosceles triangle, two of the sides have the same measure. Finally, a triangle with all three sides of the same measure is called equilateral.

Classification of triangles by side measure.

Scalene

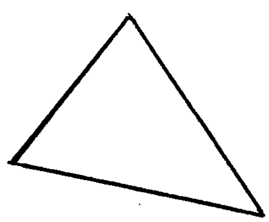


Isosceles



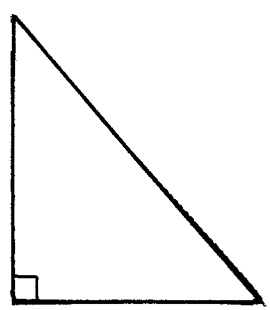
Equilateral

Triangles can also be identified by the degree measures of their interior angles (Figure 11). One of the basic theorems proved about triangles in plane geometry is that the sum of the interior angles of any plane triangle is equal to 180 degrees. Triangles can be grouped according to whether all of their angles are acute (between 0 and 90 degrees), one of their angles is a right angle (equal to 90 degrees), or, one of the angles is obtuse (greater than 90 degrees).



All angles acute

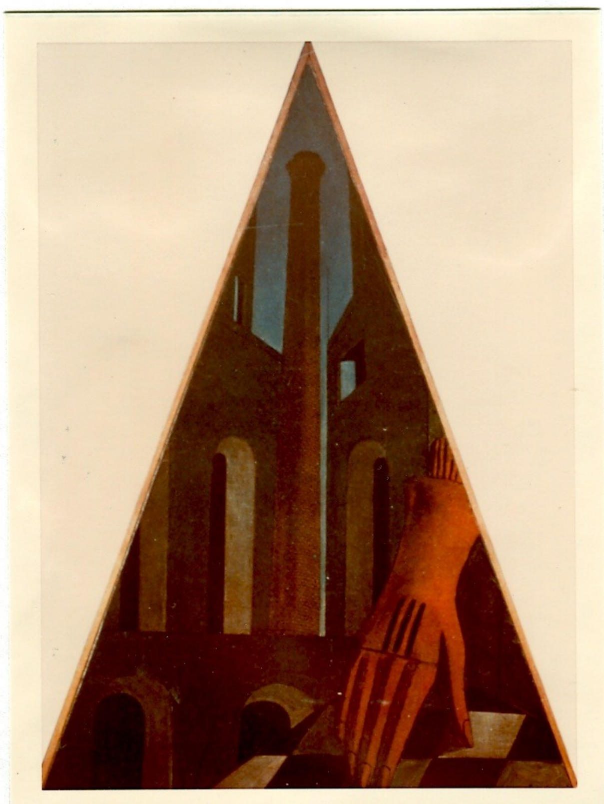
One angle obtuse



A right triangle

Triangles are rarely used to frame a painting. In The Enigma of Fatality by De Chirco (1888-1978) the mystical quality of the triangular form enhances the subject matter of this surrealistic work (Figure 12).

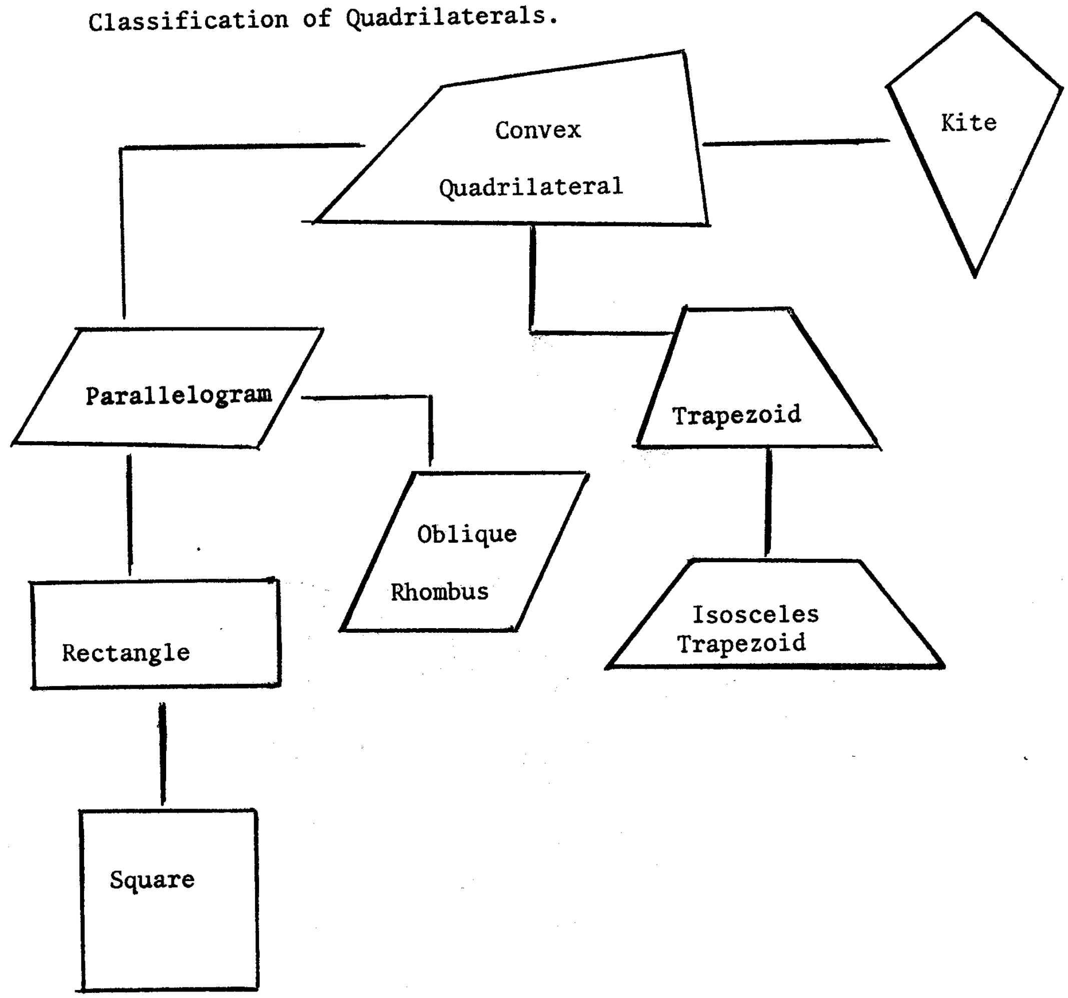
de Chirico. The Enigma of Fatality. Basel Art Museum.



Quadrilaterals

The most important polygon for dividing the artistic space into regions is the quadrilateral or four-sided polygon. Among its most frequently used forms are the square, rectangle, diamond, parallelogram, kite, and trapezoid (Figure 13).

Classification of Quadrilaterals.

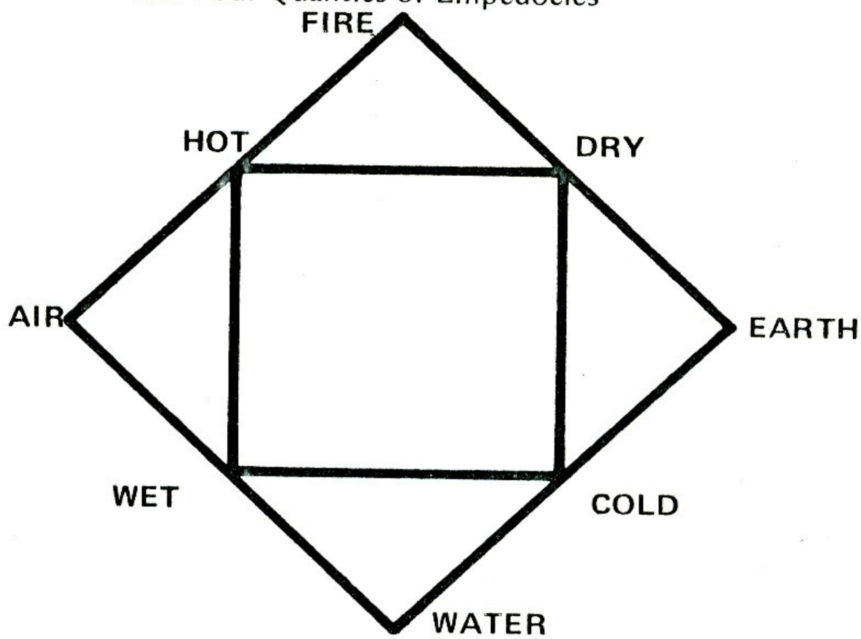


In the square and rectangle, adjacent sides are perpendicular. That is, they meet at right angles. In the rectangle the opposite sides have the same measure, while in the square all of the sides have the same measure. In the kite, pairs of adjacent sides have the same measure. The trapezoid is distinguished from the parallelogram in that only one pair of opposite sides is parallel. In an isosceles trapezoid, the two sides that are not parallel have the same measure.

Note that in the description above the phrase "have the same measure" has been used over and over again. To shorten such descriptions we resort to definitions in geometry. Henceforth, if two line segments have the same length or two angles have the same measure, we shall refer to them as being "congruent."

The quadrilateral is more closely related to the physical world as a symbol than any other figure. The Greek philosophers developed a theory of matter and sensations that was incorporated in a closed system of relationships that was reflected in a diagram by Empedocles (Figure 14). They identified four elements that constituted all matter: fire, earth, water, and air. They were arranged in the diagram as opposites; fire opposed to water and air opposed to earth. The four qualities: hot, dry, cold, and wet were arranged at the medians; hot opposed to cold and wet opposed to dry. The fact that these elements and qualities fit so neatly together in a geometrically simple relationship lent credence to this belief that dominated scientific thinking for over 2000 years. Later we shall see how Plato extended this concept to the five regular solids. Peculiar as this idea seems to our modern concepts about matter, this was a first important step in the evolution of the atomic theory. The idea that nature is made up of fundamental substances from which all else is derived is an essential step in the evolution of the theory of matter.

The Four Elements and Four Qualities of Empedocles



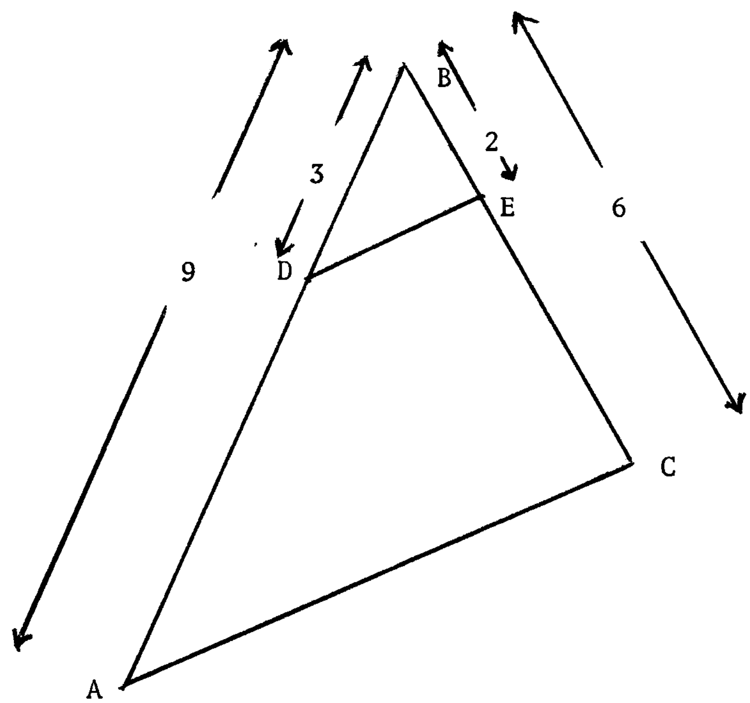
Other foursomes are linked symbolically to the quadrilateral; the four seasons, the four temperaments (choleric, sanguine, phlegmatic, and melancholic) related to the four humours: choler, blood, phlegm, and black bile that was prevalent in medieval medical theory. Religion abounds in this structure: the four gospels, the four sons of Horus. All of these find visual expression in art. In Plato's Academy, the basic training for philosophers consisted of four subjects: arithmetic (theory of numbers), geometry, astronomy, and music theory. Together they formed what is known as the quadrivium, a curriculum that persisted down to medieval times.

So much of our man-made environment is constructed in terms of rectangles and squares that it is no accident that a great deal of visual art is based upon them as well. First and foremost, the rectangle is a container, it encloses a region and sets the limits of our perceptions. Yet, because of its symmetries, it does not imply direction as the triangle does. This is one of the reasons it is identified with earthbound objects. Of all the rectangles, the square has been the greatest challenge to artists because of its simplicity and all-encompassing symmetries. Modern abstract artists, in particular, have sought to create original and personal statements within this form.

http://en.wikipedia.org/wiki/Josef_Albers

The Golden Rectangle

And yet, it is because of lack of tension that other rectangular forms were sought after. In their mastery of geometric forms, the Greeks found the one rectangle that mirrored perfectly their aesthetic philosophy: the perfect balance between reason and emotion. So precious was this discovery that they named it the "golden rectangle." It grew out of their search for ideal proportions. When we think of a proportion in arithmetic or algebra, we have in mind an equality between two ratios. For example, 2/3 = 6/9 is a proportion reflecting the equivalence of two fractions. The Greeks thought about this proportion geometrically. They regarded the numbers as lengths and envisioned the relationship as such. In Figure 15 triangles ABC and DBE are drawn so that DE is parallel to AC. By the basic proportionality theorem, if BD is 3 units, BE is 2 units, and BA is 9 units, it must follow that BC is 6 units.


B
3
2
E
6
9
D
C
A

The construction of the golden rectangle begins with the following problem. Given a line segment, to divide it into mean and extreme ratio. Figure 16 shows how the different parts of a proportion are described as "extremes" and "means." For any proportion, the product of the means is equal to the product of the extremes. Thus, in the example, 2/3 : 6/9 because the product of 2 and 9 (the extremes) is equal to the product of 3 and 6 (the means).

$$\frac{2}{3} = \frac{6}{9}$$

$$\begin{array}{lcccl} \text{extreme} & \rightarrow & \frac{a}{b} = \frac{c}{d} & \leftarrow & \text{mean} \\ \text{mean} & \rightarrow & & \leftarrow & \text{extreme} \end{array}$$

means

$$a : b :: c : d$$

extremes

$$\frac{a}{b} = \frac{c}{d} \text{ because } ad = bc$$

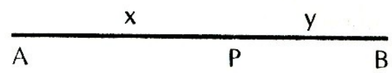
Example: $\frac{2}{3} = \frac{6}{9}$ because $2 \times 9 = 3 \times 6$

Figure 17 shows the problem in graphic and literal (algebraic) form. For a line segment to be divided into extreme and mean ratio, a point must be found, P, that divides the segment to give the proportion shown. In words, the ratio of the total segment is to the larger part as the ratio of the larger part is to the smaller part. You can see how algebraic symbols cut down the verbiage.

From the law of proportions, the product of the extremes $(x+y)y$ is equal to the product of the means (x) (x) or x^2. So, in trying to find point P, the problem is resolved into finding an equality between two products. To the Greeks this suggested a solution in terms of areas.

First, (Figure 17), a square is constructed on the given line segment AB, producing square ABCD. Then bisecting the side AD at E, the line segment EB is drawn. Next, a compass is opened to length EB with E as center and an arc is swung so that segment DA can be extended to point F. Then with AF as a side construct square AFGP. P is the point desired. But, it isn't enough to say so, it must be proven. To help, the lengths are labeled and the areas to be proven equal are shaded, the areas x^2 and $(x+y)y$. (Figure 18). A simple algebraic proof can be obtained by assuming the Pythagorean theorem is true.

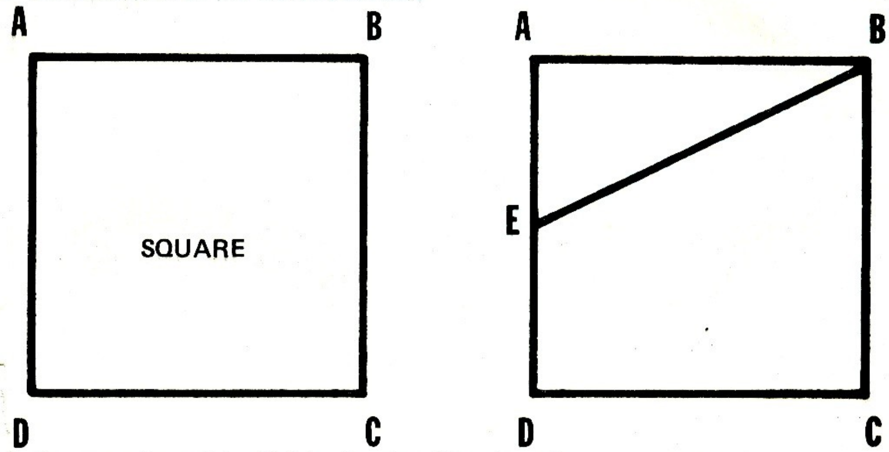
Division of a Line Segment into Extreme and Mean Ratio (The Golden Section)



$$\frac{AB}{AP} = \frac{AP}{PB} \quad \text{OR} \quad \frac{x + y}{x} = \frac{x}{y}$$

If $\frac{x + y}{x} = \frac{x}{y}$ then $(x + y)\ y = (x)(x) = x^2$

Construction of the Golden Section



Construction of the Golden Section (Continued)

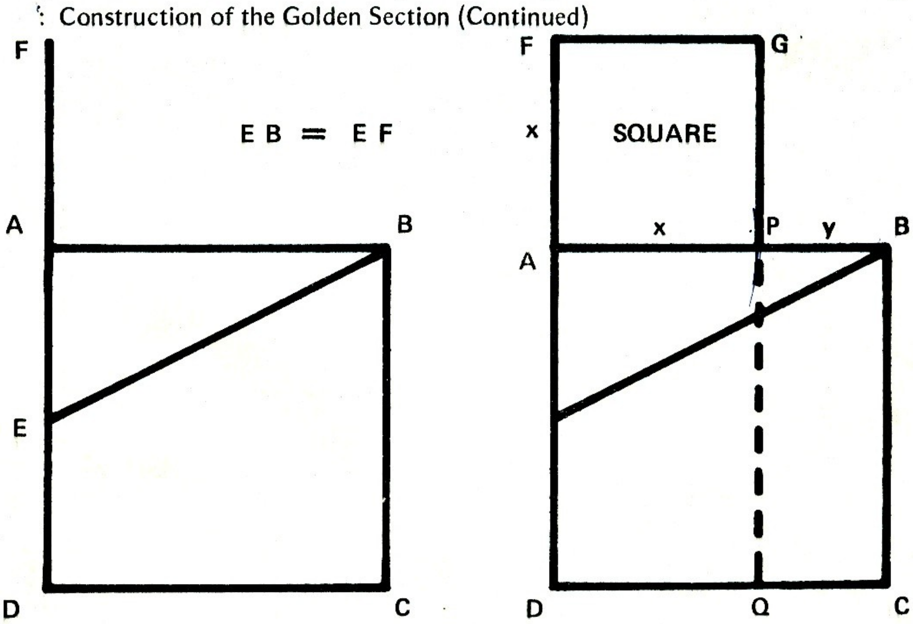
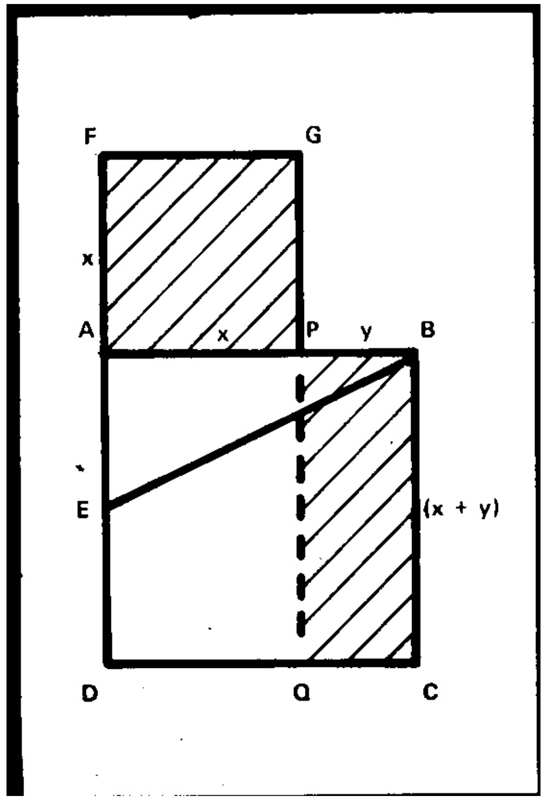
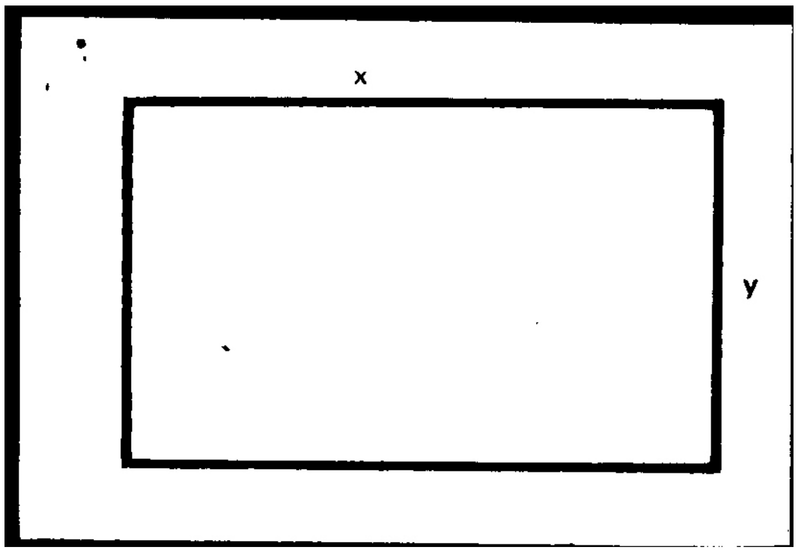


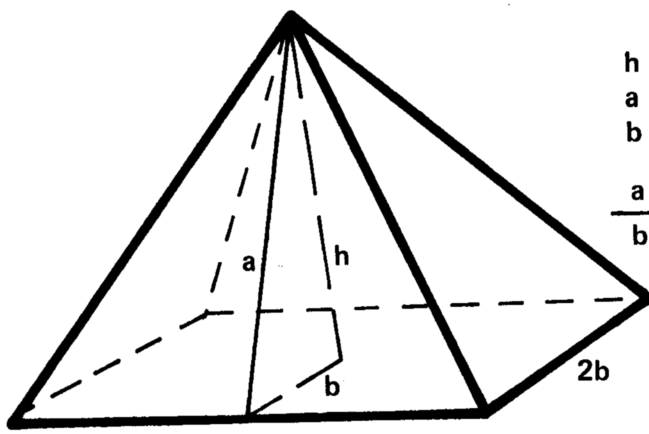
Figure 18



The division of AB into extreme and mean ratios is called "the golden section" and it is a simple step from the golden section to the golden rectangle. From Figure 18, if x is taken as a length and y as a width, a golden rectangle can be formed (Figure 19).



The golden rectangle and golden section were used in art long before the Greeks examined their mathematical properties. It is believed to have played an important part in the structure of the Pyramid of Khufu (c. 2650 B.C.). According to a passage in Herodotus, this Great Pyramid at Giza was constructed to relate the altitude, slant height, and side of the base in the golden section.[5] Modern measurements of the excavated base have confirmed this within acceptable margins of error (Figure 20).



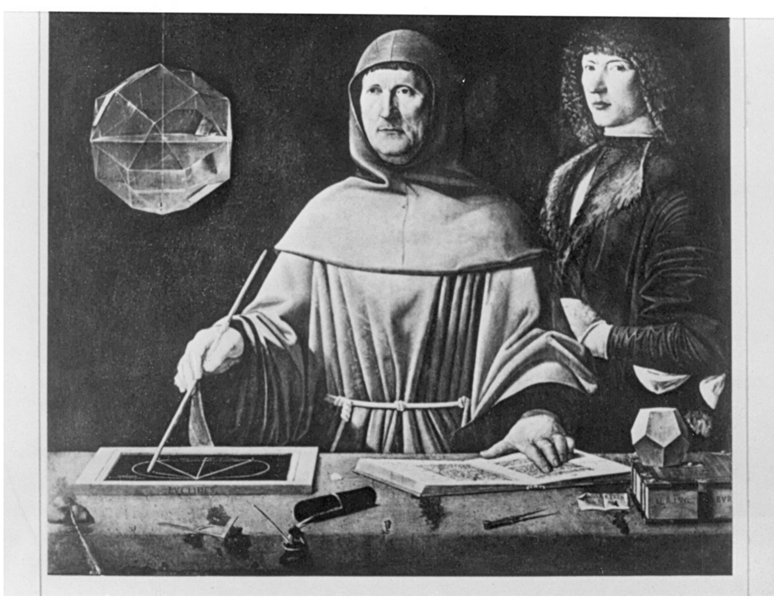
h height of pyramid
a altitude of face
b ½ of side of base

$$\frac{a}{b} = 1.6178$$

Greek vases, sculptures, and temples have also been shown to exhibit the proportions of the golden section
http://en.wikipedia.org/wiki/Parthenon

Knowledge of the golden section was passed along by artisans of the Middle Ages and its use is particularly notable in the great Gothic cathedrals.
http://en.wikipedia.org/wiki/Chartres_Cathedral

During the Renaissance, the mathematics author, Luca di Paciola (Figure 21) wrote an entire treatise about it, calling it the "Divine Proportion." Mathematically oriented artists like da Vinci and Durer learned of this proportion through their early apprenticeships and several of their works make use of it. Da Vinci produced some of the illustrations for Paciola's treatise.
http://en.wikipedia.org/wiki/Vitruvian_Man



Returning to the algebraic form of the golden sections, it is possible to derive some interesting relationships by solving the equation setting the product of the means equal to the product of the extremes: $x^2 = (x+y)y$. To realize a numerical solution to this equation, the variable y shall be set equal to 1 unit. This yields $x^2 = (x+1)(1) = x+1$. We now have a quadratic equation in a single variable. As a review, the solution will be shown through use of the quadratic formula.

Solution

Step 1: $x^2 = x + 1$ Step 2: $x^2 - x - 1 = 0$

Step 3: x =Given equation

Arranged in standard form : $Ax^2 + Bx + C = 0$

Using the quadratic formula, x = (-B ± Sqr Root (B^2 -4AC)/ 2A) ,

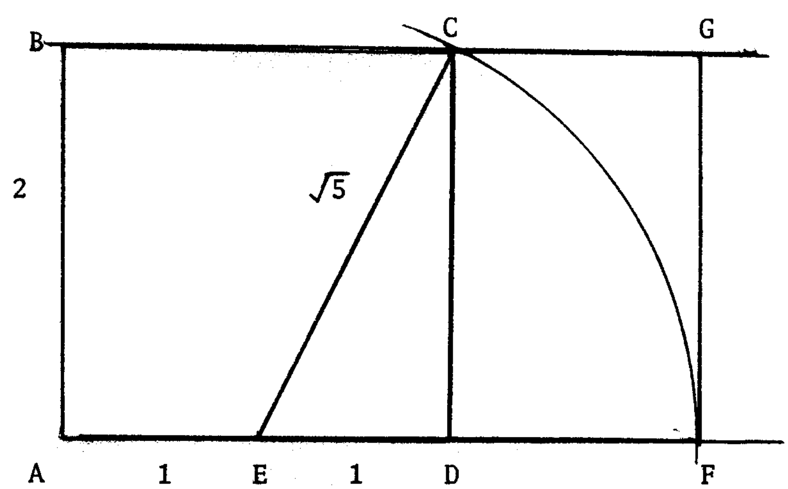
where A = 1, B = -1, C = -1

Step 4: x = (1 ± Sqr Root (1 - 4(1)(-1)) / 2(1)) = $(1 \pm \sqrt{5})/2$

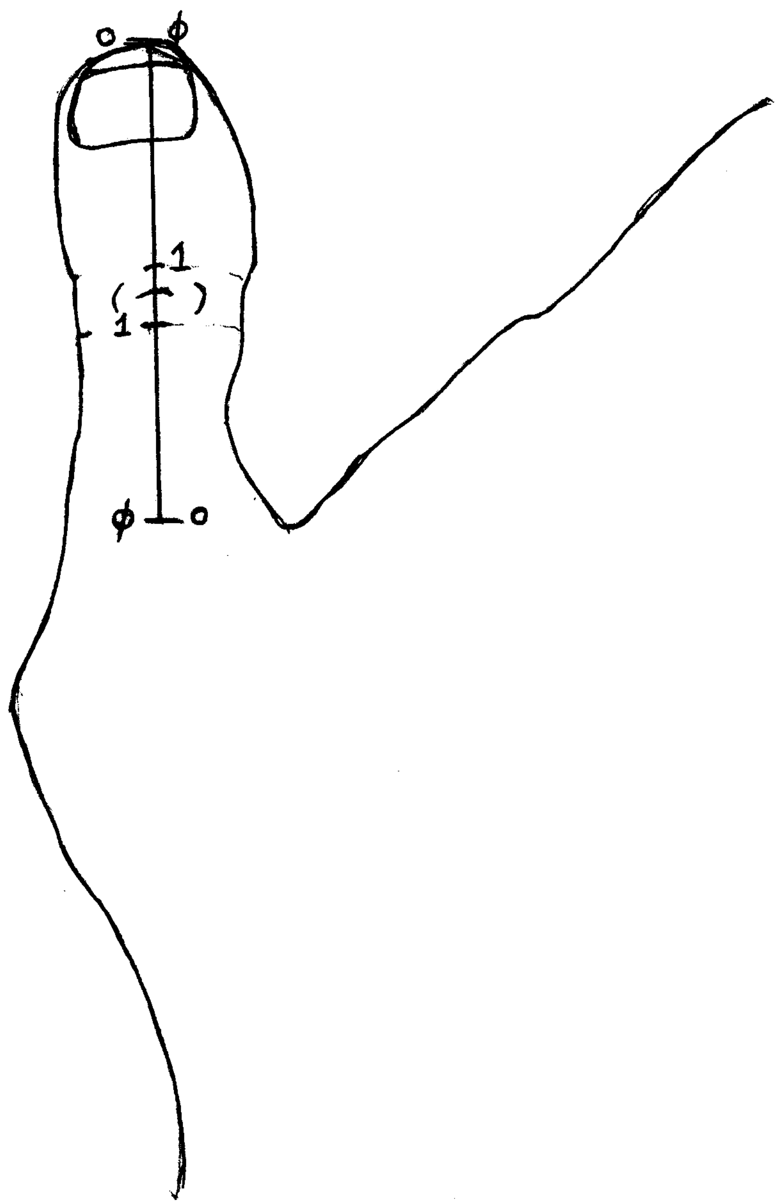
The two numerical solutions are separated and examined. Since $\sqrt{5}$ is greater than 1, one of the solutions is positive and the other is negative. For a geometric solution only the positive solution is used. We conclude then that if a line segment is divided in the golden section and the shorter segment is 1 unit in length, the longer segment is the irrational number $(1 + \sqrt{5})/2$. This number is called the "golden number" and is designated as a constant by the Greek letter "Phi," or symbolically, Ø. Earlier in the 20th century, this letter was selected in honor of the Greek sculptor, Phidias, who was reputed to have used the golden section in proportioning his lifelike sculptures of the human form.

It is but a simple step to constructing a golden rectangle illustrating the algebraic solution. In (Figure 22) a square 2 units on a side is drawn. The base AD is bisected at E and the line EC drawn. The right triangle ECD has dimensions of 1, 2, and √5. Line segment AD is extended and an arc with radius EC is drawn to intersect line AD at point F. Line BC is extended to meet the perpendicular erected from F to G. The rectangle ABGF is a golden rectangle with length equal to 1 + √5 and width equal to 2. The ratio of the length to the width (1 + √5)/2 is the golden number.
http://en.wikipedia.org/wiki/Golden_ratio

Figure 22



After the Renaissance, artistic interest in the golden section waned. Those interested in using the proportion frequently resorted to simple approximations that were expressible in terms of numbers rather than geometric constructions. For example, there is a literal "rule of thumb" that can be used (Figure 23). On a typical thumb, the length from the base to the joint is approximately related to the length from the joint to the tip in the golden section. The thumb has a long and honored history as a sighting instrument for artists.

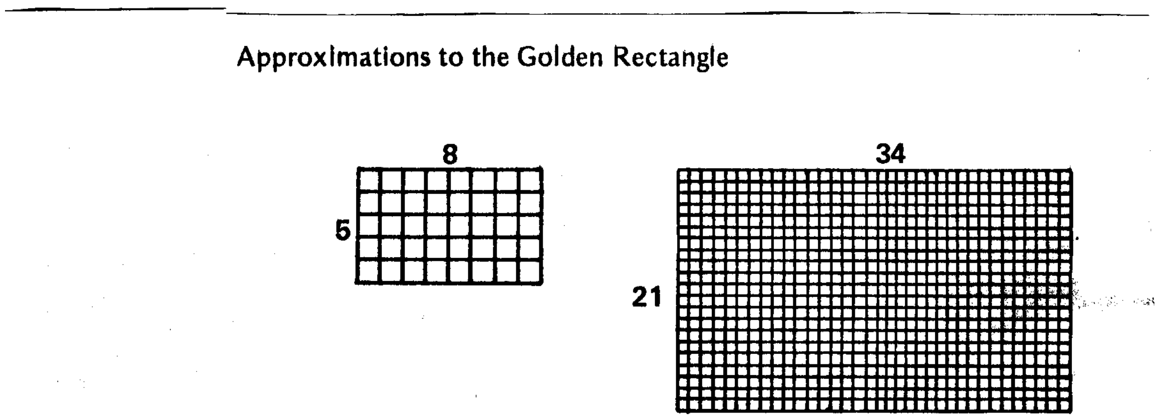

o ϕ
1
1
ϕ o

Fibonacci Sequence

Another interesting set of approximations is derived from a famous sequence of numbers attributed to the medieval mathematician, Leonardo of Pisa (c. 1180-1250). A glance at the sequence: 1,1,2,3,5,8,13,21, 34,55,89,144, ... will reveal its construction. Algebraically, this is expressed in what is called a "recursive" formula. For the Fibonacci (the mathematician's more famous name) sequence, the recursive formula is given by: $a_{\mathbf{n}} = a_{\mathbf{n-1}} + a_{\mathbf{n-2}}$

The use of the Fibonacci sequence for artistic purposes is indicated in (Figure 24). If two successive numbers like 5 and 8 are selected, with 5 serving as the width and 8 as the length, a good approximation to a golden rectangle can be obtained. The further we go in the sequence in our selections, the closer will be the approximations to a true golden rectangle. Thus, 21 for the width and 34 for the length will yield a more accurate rendition of the golden rectangle. It's interesting to note that the standard dimensions for index cards are derived from this series, particularly the 3 x 5 and the 5 x 8 cards. The relationship between the golden section and the Fibonacci sequence makes for some interesting mathematics.

Figure 24

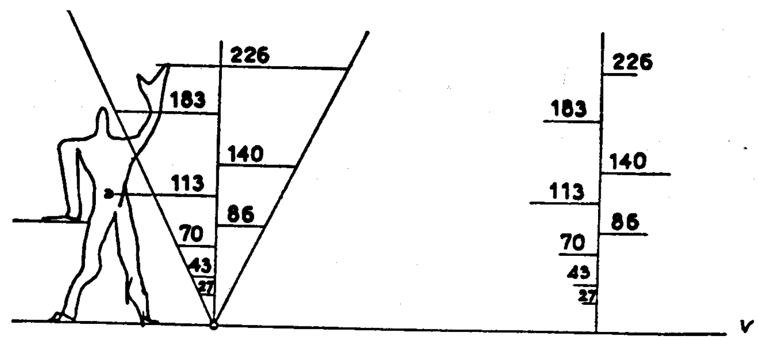


In the 19th century there was renewed interest in the proportion. A German psychologist, Fechner[9], experimentally tested the hypothesis that the golden rectangle was the most pleasing to the eye. Each subject was presented 10 different shaped rectangles and asked to choose the one that was most aesthetically satisfying. This was later confirmed by Lalo in a similar series of experiments.

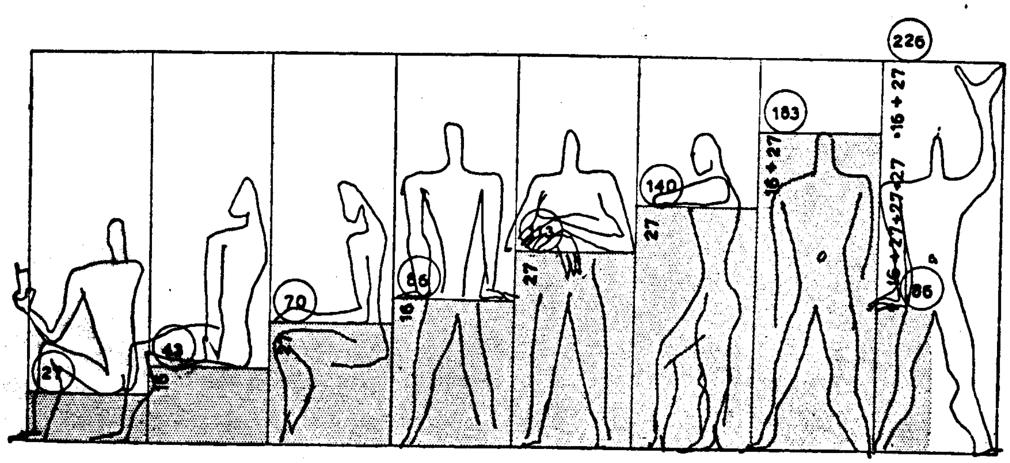
In our own century, the architect, Le Corbusier (1887-1965), was one of the most ardent proponents for the golden section. He related the golden section to human proportions in what he called "The Modulor.,,[6] This looks like a distorted human being (Figure 25).

Figure 25

Le Corbusier. The Modulor.



They may be drawn as follows:



Basically, it served as a measuring device for the artist. In the measurements a Fibonacci type sequence is built as follows: 1,5,11,16,27,43,70,113, 183. The second sequence is obtained by doubling the original: 2,10,22,32,54,86,140,226. These created various divisions that emanated from the next illustration (Figure 26) demonstrating how the square and triangle are subdivided to create the subdivisions of the figure.

Le Corbusier. Proportions obtained from the Modulor.

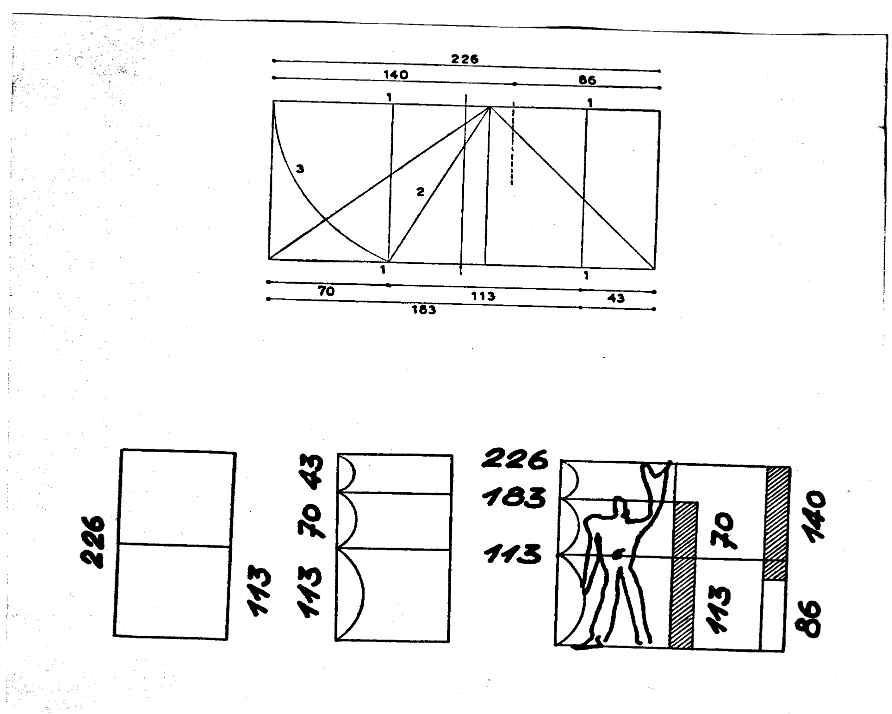


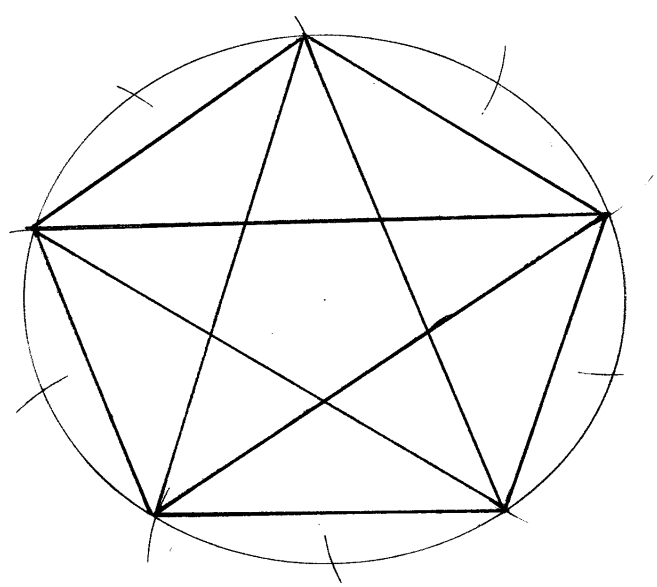
Figure 26

Le Corbusier utilized these proportions in many of his works http://en.wikipedia.org/wiki/Le_Corbusier

http://en.wikipedia.org/wiki/Notre_Dame_du_Haut

Pentagon and Pentagram

The golden section is essential for the next figure to be studied, the regular pentagon. Returning to the line segment AB divided at P in the golden section (Figure 17), if AB is the radius of a circle, then AP can be used as a side of the inscribed regular decagon or ten-sided polygon. By joining every other vertex the inscribed regular pentagon is obtained (Figure 27). Furthermore, if every other vertex of the pentagon is joined the 5-pointed star or pentagram emerges from the figure.



The pentagram was the insignia of the Pythagorean Brotherhood.

Conjured up by way of the decagon and the golden section, it gave added credence to their mystical identity of the universe symbolized by the number 10. The presence of the pentagram in Betsy Ross's design of the American flag must have particularly pleased the Founding Fathers of our country who were very much conversant with Pythagorean and Platonic philosophy. In Goethe's Faust, there is a passage where the devil, Mephistopheles, is prevented from fleeing Faust's chambers by the presence of a pentagram at the entrance.

A high point in the artistic use of the pentagon, pentagram, and hexagons was the Middle Ages. From the notebooks of Gothic artists like Villard de Honnecourt there is confirmation of the use of geometric frameworks as systems for building both pictorial and architectural designs. The human form was naturally fitted into the these figures by virtue of the head and extremities..
http://en.wikipedia.org/wiki/Villard_de_Honnecourt

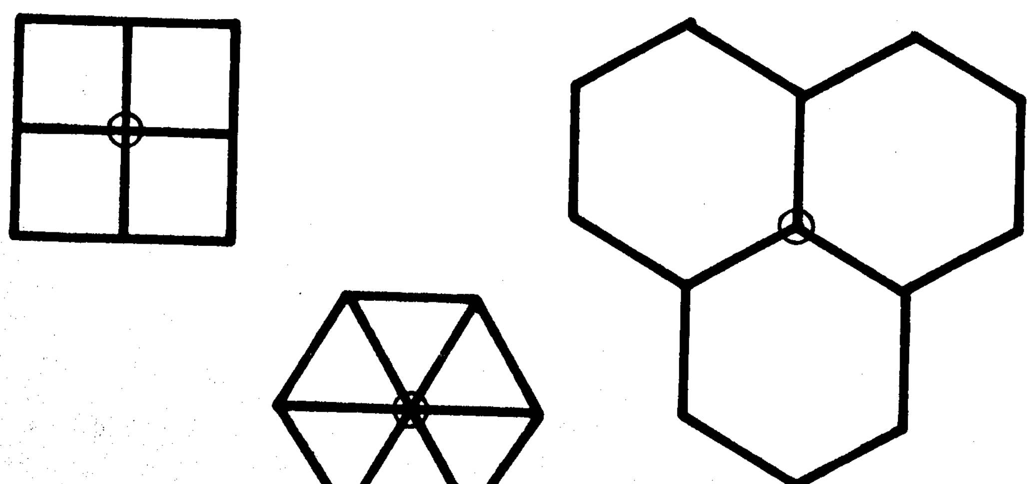
As Janson reports [7], “to the medieval artist,...drawing from life meant something far different from what it does to us—it meant filling in an abstract framework with details based on direct observation.”

Hexagons and Octagons

Regular hexagons have enjoyed a widespread application in architecture and design. They share a special property with equilateral triangles and squares, that of being able to form networks that completely enclose a point (Figure 28). For one thing this allows for their extensive use in tiling . Hexagons have also been used as frameworks for controlling the action of a painting.

Figure 28

Surrounding a Point with Regular Polygons



Both hexagons and octagons are frequently found in the prismatic

forms of towers.
http://en.wikipedia.org/wiki/Tower_of_the_Winds

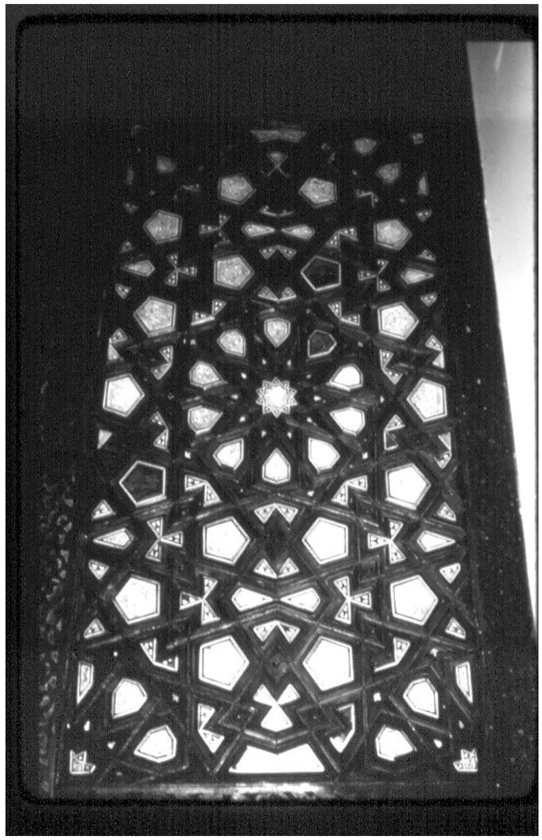
The octagon was viewed as an intermediary form between the square and the circle. As a symbol of regeneration, the octagon was also used frequently for baptismal fountains.

Other polygons occur with more rarity in artistic design, owing in part to their difficulty of construction . The city of Palmonova was laid out as a star nonogon, the projections probably serving as a defensive arrangement.

Islamic art was very rich in the combination of polygonal forms. The use of geometric forms is consistent with Muslim beliefs. As Mohammed said, "Beware of representation, whether it be of the Lord or of man, and paint nothing except trees, flowers, and inanimate objects."[8]

In his article on Ornamentation, Cirlot writes: "For Muslims, consequently, art is a kind of aid to meditation, or a sort of mandala--indefinite and interminable and opening out into the infinite, or a form of language composed of spiritual signs, or handwriting; but it can never be a mere reflection of the world of existence."[9](Figure 29).

Figure 29 (closeup of panel)



This spiritual link to the artistic use of geometric forms also motivated a devotion to pure mathematics which characterizes the high point of Islamic civilization. It's no wonder that algebra and numeration were the main contributions of Arabic mathematics.

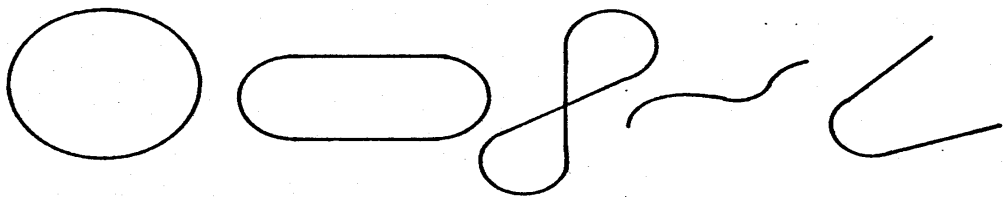
The Circle

As the number of sides of a polygon increase, the polygonal form begins to approach the shape of a circle. Theoretically, a closed regular polygon having an infinite number of sides could be used as a definition of the circle. In fact, this technique of increasing the number of sides was used by Archimedes to determine the area of a circle and to compute the value of Π. It is called "the method of exhaustion" and it is a foundation for the branch of mathematics known as the integral calculus. (Figure 30) suggests this method in an architectural realization of infinity, linked beautifully to the metaphysical idea of the "Dome of Heaven."

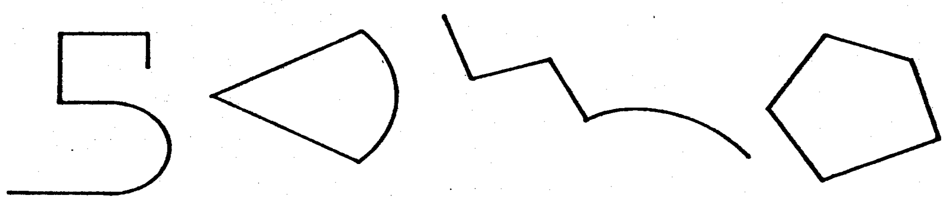
Figure 30

Mathematically, a curve is defined as the path of a moving point. In this general sense all of the polygons are curves. However, for the purpose of this section the term 'curve' will be restricted to figures which contain no polygonal parts (Figure 31).

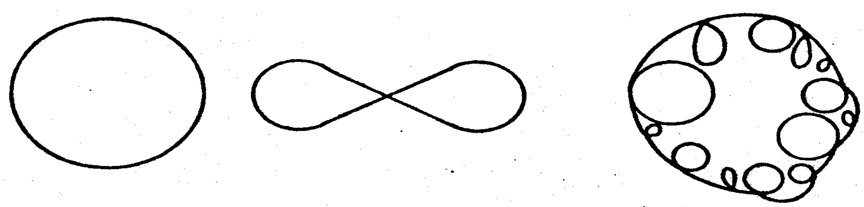
Curves



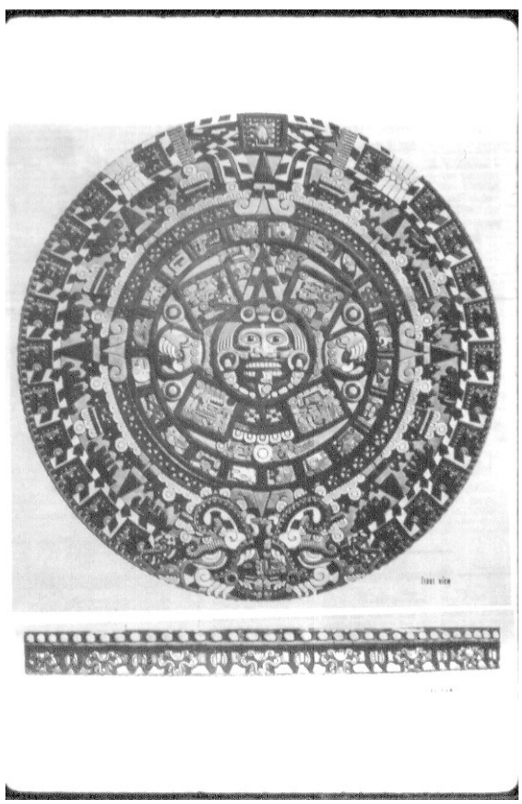
Figures Composed of Curves and Polygonal Lines



Closed Curves



A closed curve is one which can be traced so that the beginning and end point coincide. In Figure 31, only the first curve shown is a simple closed curve, since each point on the curve is only traced once. The other curves are closed, but not simple. Of all the simple closed curves, the circle has had a special significance in the history of mankind. From the earliest times it has been looked upon as the most perfect shape--"the symbol of the Self." As Aniela Jaffe has written, "Whether the symbol of the circle appears in primitive sun worship or modern religion, in myths or dreams)-:-in the mandalas drawn by Tibetan monks, in the ground plan of cities, or in the spherical concepts of early astronomers, it always points to the single most vital aspect of life--its ultimate wholeness."[10](Figure 32).



Much of the circular patterns to be seen in early Christian art go back to sun wor-shipping religions that reached a zenith in Roman times. These symbols were absorbed into Christian art and identified with the divinity of Christ. .[11]

Turning now to the circle as a structural framework in artistic organization, the ability of the circle to imply unity of composition is a product of its capacity to enclose all content within itself. The most obvious applications are the so-called "tondo" paintings and the medallions seen on Roman arches.(Figure 33)

Earlier it was shown that medieval artists used circles and polygons to guild the scaffolding of their architectural designs (Figure. This technique was of particular importance for the construction of stained glass windows . http://en.wikipedia.org/wiki/File:Canterbury_Cathedral_window_at_crossing.jpg

As Bouleau has pointed out in many examples, a circular framework for a painting has the capacity of neutralizing its rectangular boundaries and can even create the illusion of extending beyond them.

In modern art, Robert Delaunay tribute to the aviator, Louis Bleriot is captured in a panorama of circles.

Semicircles and Lunes

The semicircle is a prominent framework in Byzantine and Romanesque art, particularly in the rounded arch form used in churches (Figure 35).

Figure 35

An extension of circular arcs can be seen in figures built up of so-called "lunes," arcs of unequal radii superimposed on other forms (Figure 36). Lunes are famous in early Greek mathematics as they provided

Hippocrates of Chios (c. 460-380B.C.) with the impetus to attempt a solution to finding a square equal in area to a given circle. He was unsuccessful in this venture, but his work on squaring lunes contributed a number of important theorems that eventually became the basis for Euclid's Elements .

http://en.wikipedia.org/wiki/Lunes_of_Hippocrates

Lunes of Hippocrates of Chios with example from Exeter Cathedral.

(The lunes are the shaded regions in the diagram.)

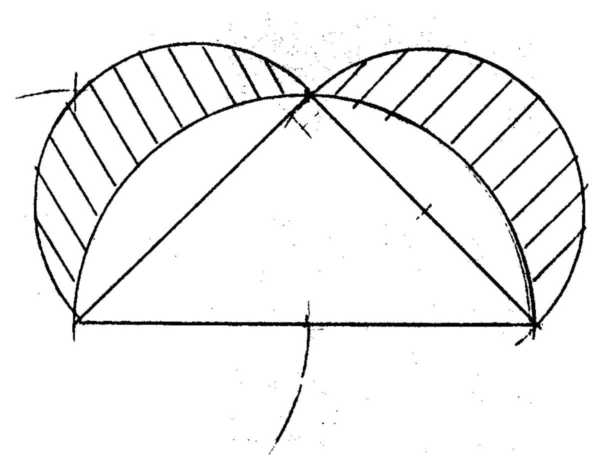
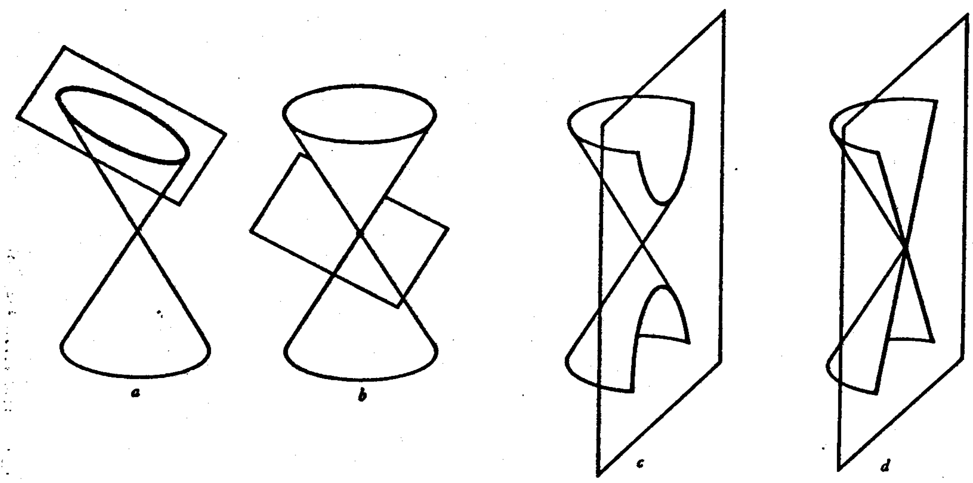


Figure 36

Conic Sections

In the branch of mathematics known as analytic geometry, the circle is grouped with a family of curves known as "conic sections." These curves are obtained, as their family name suggests, by sectioning or slicing a cone in several different ways (Figure 37). A cone consists of two infinitely radiating surfaces called "nappes" that are joined at the vertex. The right circular cone shown in Figure 37 will yield a circular base if a plane is passed through a nappe perpendicular to the axis. The axis of the cone passes through the center of the circle. If a plane is passed through either nappe at an angle between 0 and 90 degrees, the resultant section will be an ellipse. The parabola is formed by slicing a nappe parallel to a line in the surface of the cone. The hyperbola is formed with its two branches by slicing the cone with a plane parallel to the axis.

The Conic Sections.

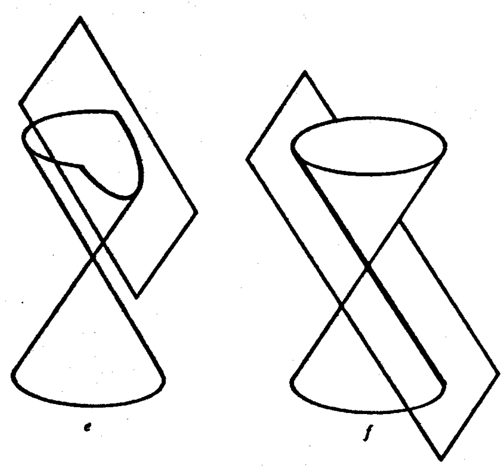


Ellipse

Single point (degenerate ellipse)

Hyperbola

Two intersecting straight lines (degenerate hyperbola)



Parabola

Single line (degenerate parabola)

Figure 37

These curves were discovered by Menaechmus around 350 B.C. while he was attempting a solution of the famous problem of doubling the volume of a given cube. The parabola and hyperbola enabled him to solve the problem geometrically. Despite the fact that these curves were theoretically examined in great detail by the mathematician, Apollonius (c. 262-190 B.C.), they were not exploited for artistic purposes except on rare occasions.

The ellipse figures prominently in the design of the Collosseum built in the first century A.D. http://en.wikipedia.org/wiki/Colloseum

The towering arch of the palace of Ctesiphon in Iraq seems to approximate a parabolic arch, but according to Fletcher[12] is an elliptical barrel vault dating from about the fourth century A.D. http://en.wikipedia.org/wiki/Ctesiphon

Interest in the conic sections did not arise until the 17th century probably as the result of Kepler's Laws for the elliptical orbits of the planets and also, the invention of analytic geometry which permitted simple geometric renditions of these curves. In any case the ellipse becomes more frequently used in the construction of domes and ornamental details.
http://en.wikipedia.org/wiki/File:Vierzehnheiligen-Basilika3-Asio.JPG

It has only been in recent times, however, that advances in structural technology have permitted the extensive use of these curves in architectural design. Most familiar to us is the parabolic arches of suspension bridges.
http://en.wikipedia.org/wiki/Golden_Gate_Bridge

Here, a basic property of the parabola is used, namely the uniform distribution of tension along the curve, making it most suitable for structures of this kind. The parabola has also been used in arches and shell structures in modern buildings.

Polar Curves

The simultaneous development of analytic geometry and analytic trigonometry led most naturally to the invention of alternate coordinate systems. The ability to transform rectangular systems into polar forms allowed for the simplification of complex algebraic equations and also enabled the graphs of these relations to be simply rendered. A beautiful example is the family of rose curves. The rose curves found in the stained glass windows of Gothic cathedrals are prime examples even though the artisans had little knowledge of the mathematics involved in the curves they created.

http://en.wikipedia.org/wiki/Rose_windows

Hyperbolic Functions

Another group of functions studied in the calculus are called hyperbolic functions. Saarinen utilized them in the construction of the gigantic westward expansion monument in St. Louis.

http://en.wikipedia.org/wiki/File:HyperbolicParaboloid.png

http://en.wikipedia.org/wiki/St_Louis_Arch

Spirals and S-Curves

A group of curves which have figured prominently in art since the beginning of man's artistic expression is the spiral. No other curve symbolizes organic growth and motion more directly. Later, in the unit on Mathematics and Nature, it will be seen how the organic world provides a bounty of spiral forms and an explanation will be given as to why this is a logical outcome of the laws of growth. In artistic ornamentation, the spiral is a basic symbol for water. It finds common application to the structure of pillars.
http://en.wikipedia.org/wiki/St._Peter%27s_Basilica#Baldacchino_and_niches

The use of the spiral to evoke the forms and actions of nature is an important part of the style of Vincent Van Gogh.
http://en.wikipedia.org/wiki/Starry_Night

One of the most interesting mathematical forms of the spiral derives from the Golden Rectangle (Figure 38). If the square on the shorter side is formed within the rectangle, it will section off another golden rectangle. Continuing this process gives rise to a sequence of smaller and smaller golden rectangles that focus in on a single point, determined by the intersection of the diagonals as shown. This limiting point becomes the eye of what is known as the logarithmic or "equiangular" spiral. The spiral is formed by noting that radii from the eye intersect the spiral at a constant angle. Later this spiral will be related to the structure of living organisms. However, craftsmen of the 16th and 17th century were particularly attracted to the shell of the Chambered Nautilus and exploited them in extravagant cup designs.

http://en.wikipedia.org/wiki/Chambered_Nautilus

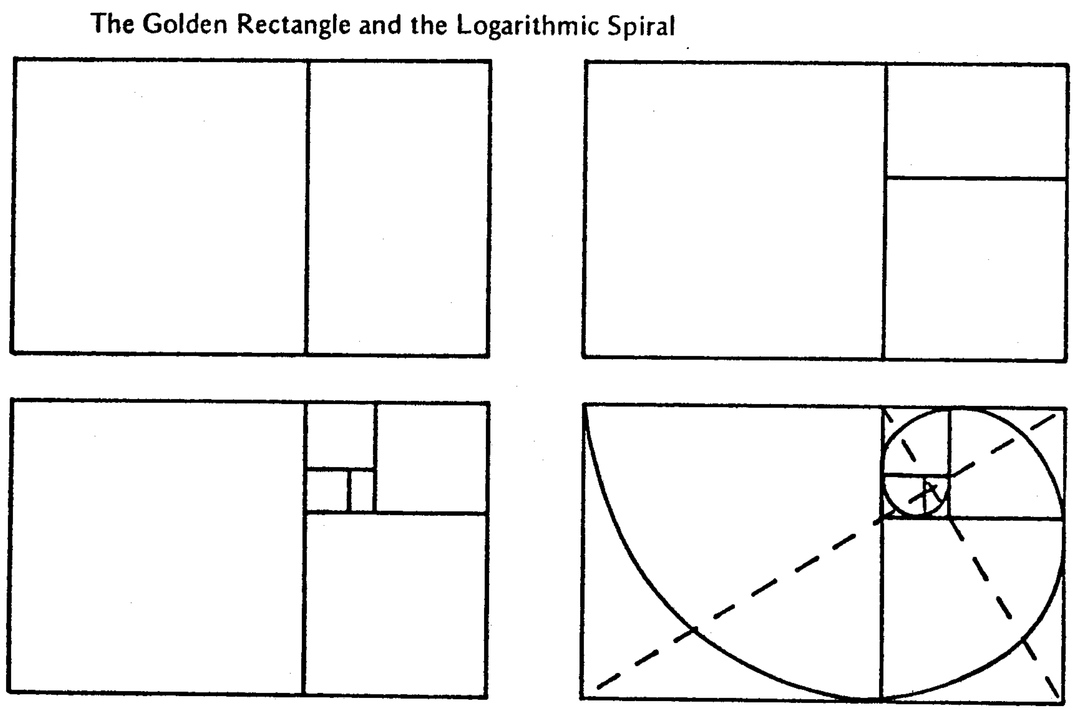


Figure 38

The spiral is closely related to the Sigma or S-curve which has been important in many artistic styles. Mathematically, this curve derives from the sine wave, the graphical interpretation of periodic motion in its simplest harmonic form. Its relationship to movement is aptly brought out by Lomazzo, the theorist of the Manneristic style: "All movements ought to be so represented that the body have the form of the serpent, to which thing nature is easily disposed."[13]

The exploitation of this compositional idea is brought to perfection in the art of Rubens and the imaginative flowing art of the Baroque period. http://en.wikipedia.org/wiki/File:Peter_Paul_Rubens_083.jpg

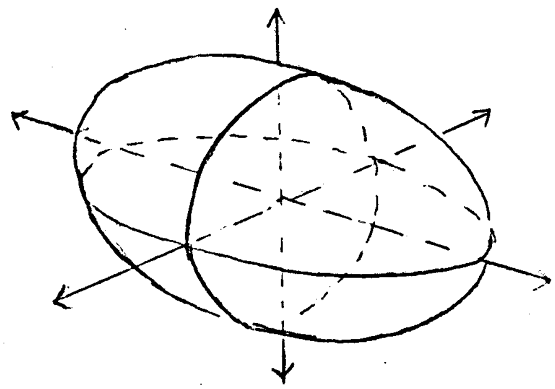
Lomazzo attributes his authority to Michelangelo who mastered the flow of the serpentine line in his magnificent sculptures. It is interesting to note that the sine wave is the ornamental symbol for air. There is an amazing primordial consistency between this symbol and Michelangelo's statement that the artist's struggle lies in "liberating the figure from the marble that imprisons it.,,[14]

Three Dimensional Forms

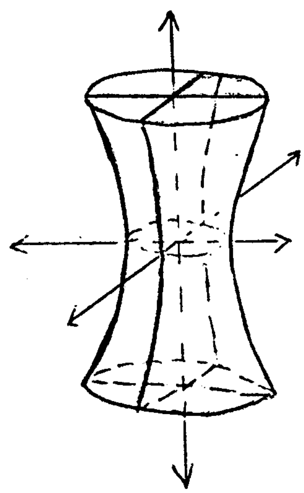
The spatial realization of the spiral is most notable in its winding about a conical or cylindrical surface. Architecturally, it has functioned best as a staircase or ramp. Sometimes the design of a whole complex of buildings can emanate from this idea.
http://en.wikipedia.org/wiki/Solomon_R._Guggenheim_Museum

The conic sections are combined in three dimensions to for m a family of figures known as quadric surfaces. (Figure 39) One such surface, known as the hyperbolic paraboloid has been extensively used in modern architecture. It has the advantage of covering wide expanses of space without requiring interior supports. http://en.wikipedia.org/wiki/TWA_Flight_Center

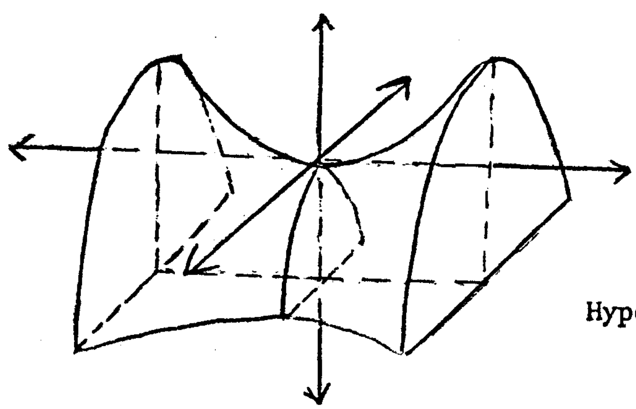
Quadric Surfaces.



Ellipsoid



Elliptic
Hyperboloid
of one sheet



Hyperbolic Paraboloid

The sphere is a rarity in architecture standing by itself. One notable example is the famous perisphere, the symbol for the 1939 New York World's Fair.
http://en.wikipedia.org/wiki/1939_New_York_World%27s_Fair

On the other hand, the hemisphere was widely used in dome construction since Roman times. Building up concentric layers of stone results in the basic dome structure that dates from about 1300 B.C..
http://en.wikipedia.org/wiki/File:Atreus-2.jpg

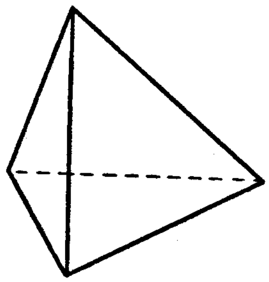
One of the most important recent developments in dome structure is the invention of the geodesic dome by Buckminister Fuller. Utilizing the fundamental geometry of polygonal frameworks, Fuller has created incredibly sturdy structures covering vast spaces. This achievement is combined with an extraordinary economy in the materials used.
http://en.wikipedia.org/wiki/Buckminister_Fuller

The cone, cylinder, and octagonal or hexagonal prism have found wide application in tower construction and are often combined.

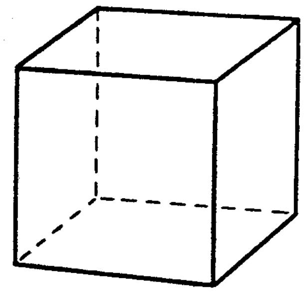
The Five Regular Solids

One of the great contributions of Greek mathematics was the determination that there exists exactly five regular solids (Figure 40). This proof is found in the last book of Euclid's Elements. The solids are described as 'regular' because each of them is composed of faces that have identical sizes and shapes. The tetrahedron is made up of four equilateral triangles, the octahedron has eight equilateral triangles, and the icosahedron twenty. The cube or hexahedron consists of six square faces and the dodecahedron is composed of twelve regular pentagons. (Figure 41) contains a set of patterns for constructing models of the regular solids.

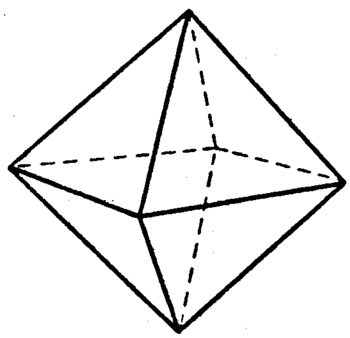
Figure 40



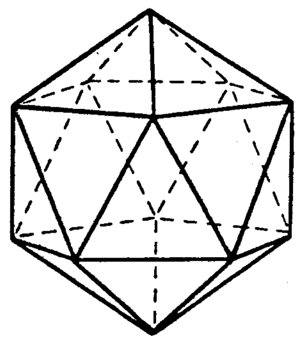
TETRAHEDRON
V = 4 F = 4 E = 6



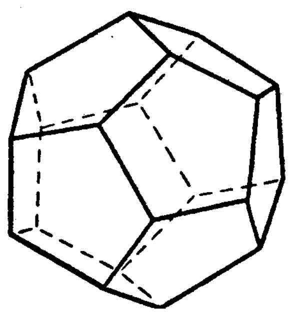
HEXAHEDRON
V = 8 F = 6 E = 12



OCTAHEDRON
V = 6 F = 8 E = 12



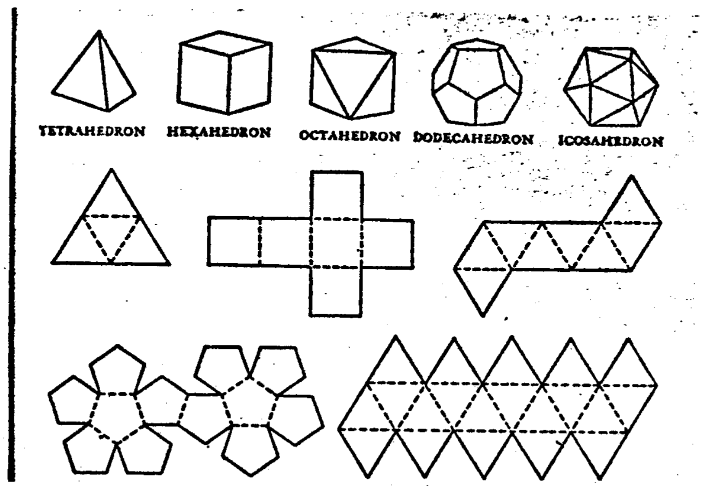
ICOSAHEDRON
V = 12 F = 20 E = 30



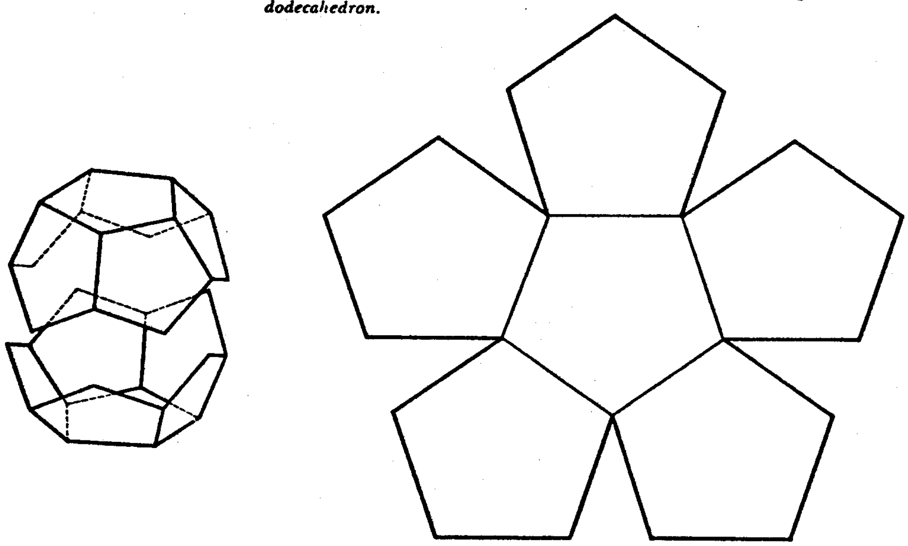
DODECAHEDRON
V = 20 F = 12 E = 30

Figure 41

The Five Regular Solids. Patterns for constructing the models.



Two caps made from this pattern fit together to form a regular dodecahedron.



Aside from the cube, the regular solids appear rarely in art or architecture. However, in Chapter 3 of this unit, the dodecahedron will be featured as a major component of Durer's famous Melancholia I .

Despite the artistic neglect of the other regular solids, a good case can be made for the cube as being the basic unit of our man-made environment. Through its extension in rectangular solids there has been established the most practical structuring of the containers we choose to live in. And yet, artistically, nothing is so confining as this most stable of all forms. From the earliest times it has been the great challenge of the architect to breathe life into this monolithic entity. The amazing variety of structural forms that evolved as a solution to this problem would recapitulate the history of architecture. It could be asked of each age how they tackled the rectangular solid, and in the answer the basic elements which determined its style could be found. From the post and lintel structure of the great temples of ancient Greece,
http://en.wikipedia.org/wiki/Theseum

to our modern hymn to the functional box, the steel girded skyscraper, there is an underlying motif of liberating its inhabitants from the rectangular confinement it implies. Even in our functionally oriented society of today there is a tendency to soften the austerity of the form.

The evolution of the rectangular solid in architecture is in itself an account of the slow and steady progress of science and technology. For example, the invention of reinforced concrete was a direct outcome of the knowledge that steel and concrete have an almost identical coefficient of expansion. By strengthening materials and making them more flexible it is

now possible to realize the most imaginative use of form in the complex architecture of today.
http://en.wikipedia.org/wiki/Wallace_Harrison

Modern sculpture has also benefited from the availability of new materials which are easier to shape. Like painting, twentieth century sculpture has moved to an abstract examination of its formal elements (See Chapter 4 on Abstraction). In the flowing surfaces of Max Bill's and Henry Moore's sculptures, there is an echo of the mathematics of topology, which is sometimes called "rubber sheet" geometry. Moore and Bill (Figure 42) raise the age old question in art that has a surprising mathematical analogy--"What is it about the human figure that remains invariant under distortion?" They respond without words in purely geometric terms.
http://en.wikipedia.org/wiki/File:HenryMoore_RecliningFigure_1951.jpg

Figure 42

The works of Pevsner take us back to where this unit began; the line and its use in artistic expression. The forms he has created seem to break the bonds of three dimensional space in almost relativistic terms. http://en.wikipedia.org/wiki/File:Pevsner-UCV.jpg

They are joined by Alexander Calder, the inventor of the mobile, in introducing the fourth dimension in art. http://en.wikipedia.org/wiki/Alexander_Calder

It is appropriate at the end of this survey of geometric forms in the visual arts to cite the development of a new medium, the computer. The results are still being questioned by art critics, yet undeniably new design patterns have been generated that have expanded the artistic imagination. There is a direct cause and effect relationship between mathematics and the artistic product resulting from computer generated forms. Yet, the human element remains. The computer artist selects the equations and relationships that eventually unfold in a graphic form. It is this freedom of selection that is really important. Art is an ordered expression of the chance elements of human existence. It integrates and organizes the random happenings of our lives to give them meaning and significance. The artist relies upon geometry only to the extent that it gives coherence to his personal statements about the human condition. http://en.wikipedia.org/wiki/Computer_art

Additional Illustrations and Comments for this Chapter

A great work of art conveys its impact through the force of its unifying idea. In achieving this communication the artist employs every means at his disposal. The conscious and intuitive blending of geometric elements is one among several of the essential tools for the realization of a compositional idea.

From ancient times, humans have enhanced their domestic existence with functional objects of artistic significance. A few of them are shown that underline the universality of geometric forms that find meaning in our lives. From the native Indians of the American southwest, we have the following:

Notice the nonogram that encircles the spiral center.

From the British Museum, some of the Elgin collection that echoes the golden age of Greece and may exhibit the golden section as well.

The Swiss are exponents of geometric form in both traditional and modern design.

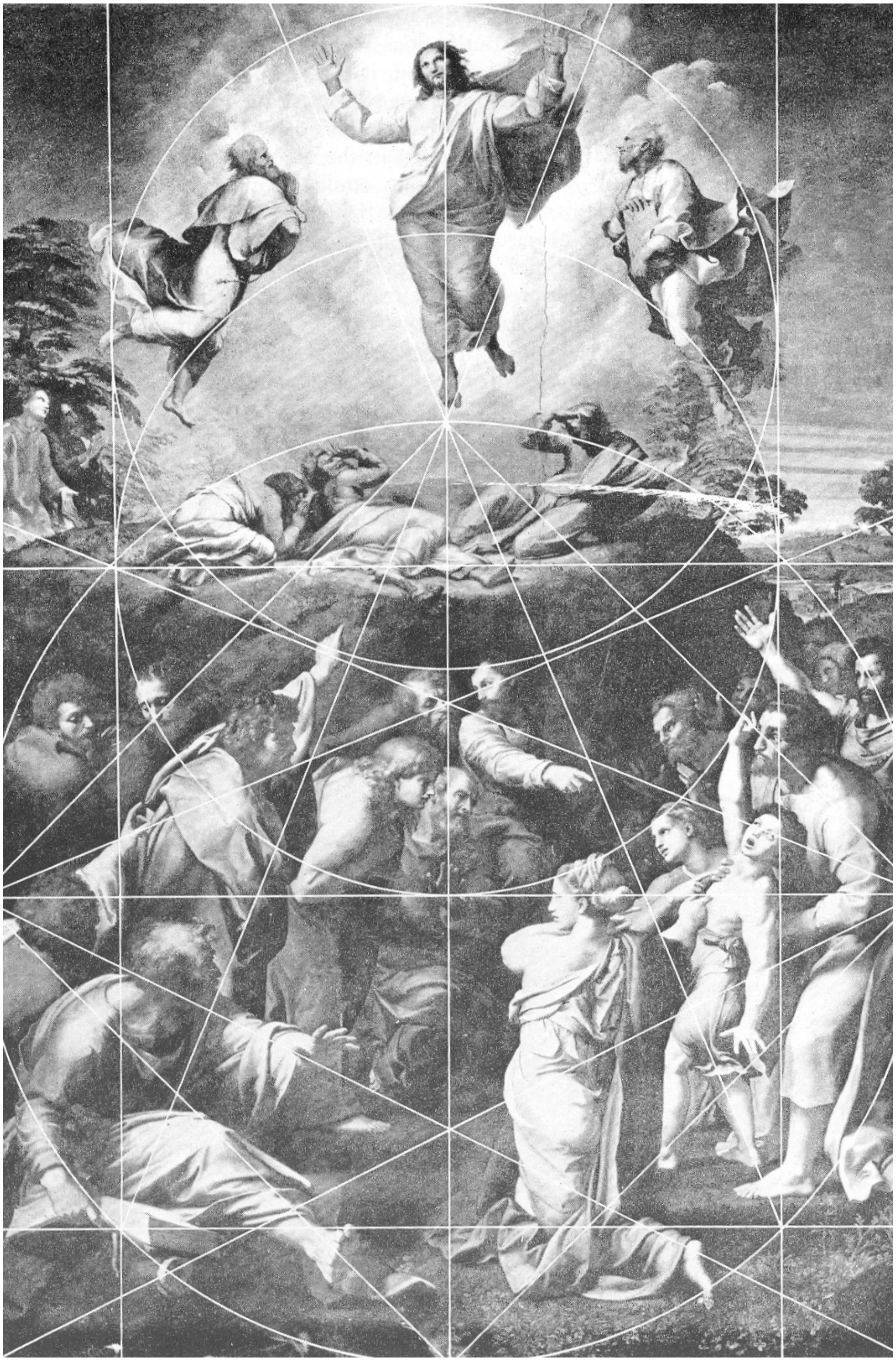
A city tower, part of several in the city of Solithurn.

Along the river bank in Zurich, 1972.

Nearby, the modern pyramid of the G.E. Building.

How better to decorate a triangle.

Throughout this presentation, Bouleau's work, The Painter's Secret Geometry has been an inspiration for much of the substance of this chapter. In the following chapter, the deep analysis of Raphael's School of Athens will continue this homage. As a final example here, I have chosen his analysis of Raphael's Transfiguration that is to be found in the Vatican. Note how the large circle seems to project itself as a model of this world as the smaller circles rise up to heaven.



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